Continuous SSB Representation of Preferences

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The theory of skew-symmetric bilinear (SSB) representation of preferences is a concise mathematical model of non-transitive decision making that has been developed in a purely algebraic setting [4, 3, 6]. Thus, the existence of a maximal preferred element in infinite-dimensional case is in jeopardy. We resolve this issue by assuming (topological) continuity of preferences.

Let P be a non-empty convex subset of a topological vector space, $\succ$ be a binary relation on P, and $\sim$ and $\succsim$ be indifference and preference-or-indifference relations defined in the standard way. The closure of $Q \subset P$ is denoted $\overline{Q}$. The *inverse* relation to $\succ$ is defined as $\{(p,q) \in P \times P : q \succ p\}$. Relation $\succ$ is *coherent* (with topology of P) if $\overline{\{q \in P : q \succ p\}} = \{q \in P : q \succsim p\}$ for all $p \in P$ such that $\{q \in P : q \succ p\} \neq \emptyset$. A coherent relation $\succ$ is *upper semi-Fishburn* if $\{q \in P : q \succ p\}$ is convex for all $p \in P$.

Theorem 1. *If P is, moreover, compact, and $\succ$ is an upper semi-Fishburn relation on P, then there exists a maximal element of P w.r.t. $\succ$.*

A binary relation is *lower semi-Fishburn* if its inverse is upper semi-Fishburn; a *Fishburn* relation is lower and upper semi-Fishburn. A binary relation $\succ$ is *balanced* if for all $p, q, r \in P$ and $0 < \lambda < 1$, it holds that $q \sim \frac{1}{2}p + \frac{1}{2}r$ and $\lambda p + (1-\lambda)r \sim \frac{1}{2}p + \frac{1}{2}q$ implies $\lambda r + (1-\lambda)p \sim \frac{1}{2}r + \frac{1}{2}q$.

Theorem 2. *A binary relation $\succ$ on P is a balanced Fishburn relation iff there exists a continuous SSB functional Φ on $P \times P$ such that for all $p, q \in P$, $p \succ q \Leftrightarrow \Phi(p,q) > 0$.*

Denote by $\mathscr{P}(X)$ the set of all (regular Borel) probabilistic measures on X, equipped with weak topology[1]. The following result stems from Theorem 1 and Theorem 2.

[1] Or, more precisely, weak* topology in the sense of functional analysis, see [1].

Theorem 3. *For a balanced Fishburn relation $\succ$ defined on a compact Hausdorff space X, and a continuous SSB representation ϕ of $\succ$, we define relation $>$ on $\mathscr{P}(X)$ by*

$$p > q \quad \text{iff} \quad \int_{X\times X} \phi(x,y)dp(x)dq(y) > 0.$$

Then, for a closed and convex set $K \subset \mathscr{P}(X)$ there exists $p \in K$ such that $p \geq q$ for all $q \in K$.

By stating the above theorem for $P = \mathscr{P}(X)$, the assumptions about relation $>$ on $P \setminus \mathscr{P}(X)$ may be omitted, c.f. [2, Theorem 5][2]. Moreover, we have shown the existence of a maximally preferred measure for any closed and convex subset of $\mathscr{P}(X)$. Thus we have generalized [2, Theorem 5] in these two aspects.

Acknowledgement

This research has been supported by grant GA17-08182S of the Czech Science Foundation. The author is grateful to F. Brandl and T. Kroupa for valuable discussions of this topic.

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[2] This theorem has been stated for a locally compact set of outcomes X, referring to a minimax theorem from [5]. There, the compactness of X is required, and so we believe that the word "locally" was most likely used unintentionally in [2]. Indeed, local compactness of X is not sufficient; consider $X = \mathbb{R}$, $P = \mathscr{P}(X)$ and $\phi \equiv 1$.