On attempts to characterize facet-defining inequalities of the cone of exact games

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Abstract

The sets of balanced, totally balanced, exact and supermodular games play an important role in cooperative game theory. These sets of games are known to be polyhedral cones. The (unique) non-redundant description of these cones by means of the so-called facet-defining inequalities is known in cases of balanced games and supermodular games, respectively. The facet description of the cones of exact games and totally balanced games are not known and we present conjectures about what are the facet-defining inequalities for these cones.

We introduce the concept of an *irreducible min-balanced* set system and conjecture that the facet-defining inequalities for the cone of totally balanced games correspond to these set systems. The conjecture concerning exact games is that the facet-defining inequalities for this cone are those which correspond to irreducible min-balanced systems on strict subsets of the set of players and their conjugate inequalities. A consequence of the validity of the conjectures would be a novel result saying that a game m is exact if and only if m and its reflection are totally balanced.

1 Introduction: former results overview

Important classes of set functions used as mathematical models in the coalition game theory are: the class of *balanced games* $\mathcal{B}(N)$, the class of *totally balanced games* $\mathcal{T}(N)$, the class of *exact games* $\mathcal{E}(N)$ and the class of *supermodular games* $\mathcal{S}(N)$, named traditionally *convex* in game-theoretical community. One has

$$\mathcal{B}(N) \supseteq \mathcal{T}(N) \supseteq \mathcal{E}(N) \supseteq \mathcal{S}(N)$$

and it is well-known that all these sets are polyhedral cones in the space $\mathbb{R}^{\mathcal{P}(N)}$, where $\mathcal{P}(N) = \{A : A \subseteq N\}$ is the power set of the set of players N. That means that each of the cones can be specified by finitely many linear inequalities.

These set functions occur in other contexts, for example, in the context of *imprecise probabilities*. More specifically, supermodular games correspond to 2-monotone lower probabilities, exact games to coherent lower probabilities and balanced games to lower probabilities avoiding sure loss [9].

Recall that every full-dimensional polyhedral cone K in an Euclidean space has uniquely determined set of the so-called *facet-defining* inequalities, where the uniqueness of each inequality is up to a positive multiple. Specifically, these inequalities determine proper faces of K of maximal dimension, called *facets*. The complete list of facet-defining inequalities then provides the least possible inequality description of K, unique up to positive multiples.

The above cones $\mathcal{B}(N), \ldots, \mathcal{S}(N)$ are not full-dimensional in $\mathbb{R}^{\mathcal{P}(N)}$ but adding a one-dimensional linear space $\mathcal{C}(N)$ of constant functions turns them into full-dimensional cones $B(N), \ldots, S(N)$. The facet-defining inequalities of these extended cones then induce a non-redundant inequality description of the original cones of games. The non-redundant inequality description of these cones is known in cases of supermodular and balanced games only.

The facets of the *supermodular* cone $S(N)$ are defined by inequalities of the form $m(\{i,j\} \cup L) + m(L) - m(\{i\} \cup L) - m(\{j\} \cup L) \geq 0$ for $m \in S(N)$, where $L \subset N$ and $i, j \in N \setminus L$ are distinct [6]; these inequalities were known to correspond to elementary conditional independence statements [13].

The non-redundant inequalities for the cone $\mathcal{B}(N)$ of *balanced* games were characterized by Shapley [12] on basis of former results by Bondareva [1]. These inequalities correspond to "minimal balanced collections" of subsets of N whose union is N; in this paper we call such collections *min-balanced systems* on N.

The consequence of the fact that $\mathcal{B}(N)$ is a polyhedral cone is the observation that the set of totally balanced games $\mathcal{T}(N)$ is a polyhedral cone. Nonetheless, as far as we know, the facet-defining inequalities for the cone $\mathcal{T}(N)$ have not been described/discussed in the literature.

Recently, the fact that set of exact games $\mathcal{E}(N)$ forms a convex cone has been derived [3]. Shortly after that Lohman *et al.* [8] even showed that the set of exact games is a polyhedral cone. Specifically, the exact games were characterized by means of finitely many linear inequalities that correspond to the so-called "minimal exact balanced" collections of subsets of N. Although finitely many linear inequalities specifying $\mathcal{E}(N)$ were classified in [8], many of these inequalities are already known to be redundant.

In this paper we mainly deal with the question of what are the facet-defining inequalities for the exact cone. On basis of our own computations as well as computations made by Quaeghebeur in connection with his thesis [11] we found and classified these inequalities in case $|N| \leq 5$. We analyzed the results and revealed certain symmetry in the problem. More specifically, the facet-defining inequalities for $E(N)$ come in pairs: every such inequality is accompanied with a *conjugate* one. We have shown that this is a consequence of the fact that the cone $E(N)$ is closed under a special *reflection* transformation.

We even came to sensible conjectures about what are the facet-defining inequalities for the cones $T(N)$ and $E(N)$. The basis of them is the concept of a *min-balanced system* on $M \subseteq N$, where $|M| \geq 2$, which is a certain collection of subsets of M. Special *irreducible min-balanced systems* seem to play the crucial role. The conjecture concerning the totally balanced cone is that these irreducible systems correspond to facets of $T(N)$. The conjecture concerning the exact cone is that every facet-defining inequality for $E(N)$ is either given by an irreducible min-balanced system on some strict subset $M \subset N$, $|M| \geq 2$, or it a conjugate inequality to such an inequality.

We also briefly report on our effort to develop a computer programme for generating all (permutational types of) min-balanced systems and irreducible min-balanced systems. We employed the algorithm by Peleg [10] and reformulated the problem in terms special bipartite graphs using the BLISS algorithm by Junttila and Kaski [5].

2 Basic concepts and facts

Throughout the text N is a finite set of *players* such that $|N| \geq 2$. Given $S \subseteq N$, the symbol χ_S will denote zero-one incidence vector of S (in $\mathbb{R}^N$); that is, $\chi_S(i) = 1$ if $i \in S$ and $\chi_S(j) = 0$ if $j \in N \setminus S$.

A *game* is a set function $m : \mathcal{P}(N) \to \mathbb{R}$ such that $m(\emptyset) = 0$. The *core* of a game m is a polytope (= bounded polyhedron) in $\mathbb{R}^N$ given by

$$C(m) \;:=\; \{\, [x_i]_{i\in N} \;:\; \sum_{i\in N} x_i = m(N) \quad \& \quad \forall\, S \subseteq N \quad \sum_{i\in S} x_i \geq m(S) \,\}\,.$$

A game m is *balanced* if it has a non-empty core: $C(m) \neq \emptyset$; it is called *totally balanced* if, for each $M \subseteq N$, $|M| \geq 2$, the restriction of m to $\mathcal{P}(M)$ is balanced. A balanced game is called *exact* if every lower bound is tight:

$$\forall\, S \subseteq N \quad \exists\, x \in C(m) \qquad \sum_{i\in S} x_i = m(S)\,;$$

an equivalent definition is that m can be reconstructed from its core by minimization: $m(S) = \min\{\, \sum_{i\in S} x_i \;:\; x \in C(m)\}$ for any $S \subseteq N$. Well-known facts are that every exact game is totally balanced and that every supermodular game is exact [4].

Definition 1 A system $\mathcal{B} \subseteq \mathcal{P}(N)$ is *min-balanced on* a non-empty set $M \subseteq N$ if it is a minimal set system such that χ_M is in the conic hull of $\{\chi_S \,:\, S \in \mathcal{B}\}$.

Of course, the minimality is meant in sense of inclusion of set systems.

Lemma 2 A set system $\mathcal{B} \subseteq \mathcal{P}(N)$ is min-balanced on $\emptyset \neq M \subseteq N$ if and only if the following two conditions hold:

(i) there exist strictly positive coefficients $\lambda_S > 0$ for $S \in \mathcal{B}$ such that

$$\chi_M = \sum_{S \in \mathcal{B}} \lambda_S \cdot \chi_S \qquad \text{where } M = \bigcup \mathcal{B}, \text{ and}$$

(ii) the incidence vectors $\{\chi_S \in \mathbb{R}^N : \; S \in \mathcal{B}\}$ are linearly independent.

Hence, $\mathcal{B} \subseteq \mathcal{P}(N)$ is min-balanced iff it is a minimal set system satisfying (i).

The condition (i) is the *balancedness* condition from [12]; (ii) is equivalent to minimality and implies the uniqueness of the coefficients λ_S in (i).

Proof. To show the necessity of (i) write $\chi_M = \sum_{S \in \mathcal{B}} \lambda_S \cdot \chi_S$ with $\lambda_S \geq 0$. If λ_S vanishes for some S then we take $\mathcal{B}' = \{T \in \mathcal{B} : \lambda_T > 0\}$ to get a contradictory conclusion that $\mathcal{B}'$ is a strict subsystem of $\mathcal{B}$ satisfying the requirement. The necessity of (ii) can then be shown by a contradiction: otherwise a non-vanishing system of coefficients $\{\gamma_S : S \in \mathcal{B}\}$ exists such that $\sum_{S \in \mathcal{B}} \gamma_S \cdot \chi_S = \mathsf{0} \in \mathbb{R}^N$. For any $\varepsilon \geq 0$ put $\lambda_S^\varepsilon := \lambda_S + \varepsilon \cdot \gamma_S$ and consider $\chi_M = \sum_{S \in \mathcal{B}} \lambda_S^\varepsilon \cdot \chi_S$. Since all λ_S are strictly positive, maximal $\varepsilon > 0$ exists such that λ_S^ε are all non-negative. Put $\mathcal{B}' = \{T \in \mathcal{B} : \lambda_T^\varepsilon > 0\}$ and derive the contradiction analogously.

Conversely, if both (i) and (ii) holds then $\chi_M = \sum_{S \in \mathcal{B}} \lambda_S \cdot \chi_S$ with $\lambda_S > 0$. Assume for a contradiction that $\mathcal{C} \subset \mathcal{B}$ exists such that $\chi_M = \sum_{S \in \mathcal{C}} \nu_S \cdot \chi_S$ with $\nu_S \geq 0$, $S \in \mathcal{C}$. Put $\nu_S = 0$ for $S \in \mathcal{B} \setminus \mathcal{C}$. Then $\mathsf{0} = \sum_{S \in \mathcal{B}} (\lambda_S - \nu_S) \cdot \chi_S$, which contradicts (ii). The last claim is easy to derive from the former one. $\square$

We intentionally restrict our attention to *non-trivial* min-balanced systems $\mathcal{B}$ with $|\mathcal{B}| \geq 2$; each such a system is ascribed the following inequality

$$m(\bigcup \mathcal{B}) - \sum_{S \in \mathcal{B}} \lambda_S \cdot m(S) \geq 0 \tag{1}$$

in which variables are represented by $m(S)$, $S \subseteq N$. We have shown in [7, Observation 4] that any non-trivial min-balanced system $\mathcal{B}$ on M satisfies the following conditions:

- the intersection $\bigcap \mathcal{B}$ is empty, one has $\emptyset, M \notin \mathcal{B}$, $|M| \geq 2$, and
- there are at most $|M|$ sets in $\mathcal{B}$.

The result from [12] is as follows.

Proposition 3 *The facet-defining inequalities for the cone $\mathcal{B}(N)$ are just the inequalities* (1) *for non-trivial min-balanced systems $\mathcal{B}$ on N.*

3 Conjugate inequalities

To reveal some important symmetry in the problem, it is suitable to consider the space $\mathbb{R}^{\mathcal{P}(N)}$ and extend all considered cones $\mathcal{B}(N),\ldots,\mathcal{S}(N)$ to this space. Formally, for $m\in\mathbb{R}^{\mathcal{P}(N)}$, a shifted function $\widetilde{m}$ given by $\widetilde{m}(S):=m(S)-m(\emptyset)$ for $S\subseteq N$ is a game and one can define:

$$
\begin{aligned}
B(N) &:= \{\, m\in\mathbb{R}^{\mathcal{P}(N)} \;:\; \widetilde{m} \text{ is a balanced game}\,\},\\
T(N) &:= \{\, m\in\mathbb{R}^{\mathcal{P}(N)} \;:\; \widetilde{m} \text{ is a totally balanced game}\,\},\\
E(N) &:= \{\, m\in\mathbb{R}^{\mathcal{P}(N)} \;:\; \widetilde{m} \text{ is an exact game}\,\},\\
S(N) &:= \{\, m\in\mathbb{R}^{\mathcal{P}(N)} \;:\; \widetilde{m} \text{ is a supermodular game}\,\}.
\end{aligned}
$$

All these cones are full-dimensional in $\mathbb{R}^{\mathcal{P}(N)}$ and their shared linearity space appears to be the linear space of *modular* functions

$$L(N):=\{m\in\mathbb{R}^{\mathcal{P}(N)} : m(C\cup D)+m(C\cap D)=m(C)+m(D) \ \text{ for } C,D\subseteq N\},$$

which has the dimension $1+|N|$; see [7, § 1.1].

The task to find/characterize facets of the original cones $\mathcal{B}(N),\ldots,\mathcal{S}(N)$ of games appears to be equivalent to finding facets of the above extended cones. Some geometric considerations lead to the conclusion that every facet-defining inequality for such a full-dimensional cone K in $\mathbb{R}^{\mathcal{P}(N)}$ has the form

$$\sum_{S\subseteq N}\alpha(S)\cdot m(S)\geq 0 \qquad \text{for } m\in\mathbb{R}^{\mathcal{P}(N)}, \tag{2}$$
$$\text{where } \textstyle\sum_{S\subseteq N}\alpha(S)=0,\ \sum_{S\subseteq N:i\in S}\alpha(S)=0 \text{ for any } i\in N,$$

and the coefficients $\alpha(S)$, $S\subseteq N$, are rational numbers. Thus, without loss of generality, we can multiply (2) by a positive factor to obtain relatively prime integers as coefficients. This is a *standardized form* of the inequality (2).

An inequality of the form (2) for $m\in B(N),\ldots,S(N)$ can be identified with the inequality $\sum_{\emptyset\neq S\subseteq N}\alpha(S)\cdot\widetilde{m}(S)\geq 0$ for $\widetilde{m}\in\mathcal{B}(N),\ldots,\mathcal{S}(N)$: for the inverse relation put $\alpha(\emptyset):=-\sum_{\emptyset\neq S\subseteq N}\alpha(S)$; for details see [7, § 1.1]. In particular, the inequality (1) has an extended version

$$m(\bigcup\mathcal{B})-\sum_{S\in\mathcal{B}}\lambda_S\cdot m(S)+(-1+\sum_{S\in\mathcal{B}}\lambda_S)\cdot m(\emptyset)\geq 0\,.$$

The point is that the considered cones, except for $T(N)$, are closed under the following linear self-transformation of $\mathbb{R}^{\mathcal{P}(N)}$. By a *reflection* of $m\in\mathbb{R}^{\mathcal{P}(N)}$ we mean $m^*\in\mathbb{R}^{\mathcal{P}(N)}$ given by

$$m^*(T):=m(N\setminus T) \qquad \text{for any } T\subseteq N.$$

It is nothing but inner composition with the “complement” mapping.

The inequality (2) can be ascribed a *conjugate inequality* of the form

$$\sum_{T\subseteq N} \alpha^*(T)\cdot m(T) \geq 0 \qquad \text{where} \;\; \alpha^*(T) := \alpha(N\setminus T) \;\; \text{for any } T\subseteq N, \qquad (3)$$

required for $m \in \mathbb{R}^{\mathcal{P}(N)}$. An important observation appears to be the equality

$$\sum_{T\subseteq N} \alpha^*(T)\cdot m(T) = \sum_{T\subseteq N} \alpha(N\setminus T)\cdot m^*(N\setminus T) = \sum_{S\subseteq N} \alpha(S)\cdot m^*(S)\,,$$

which easily implies that, whenever (2) is valid for vectors in a cone K which is closed under reflection, then (3) is valid for vector in K, and, of course, vice versa. In fact, one of our theoretical results is that (2) is facet-defining for K closed under reflection iff (3) is facet-defining for K [7, Lemma 26].

Every inequality of the form (2) defines a set system

$$\mathcal{B}_\alpha \;:=\; \{S\subseteq N \,:\, \alpha(S)<0\}\,. \qquad (4)$$

Our analysis of facet-defining inequalities for $E(N)$ in case $|N|\leq 5$ written in the form (2) revealed that every system $\mathcal{B}_\alpha$ is either min-balanced on $M\subset N$ or it is *conjugate* to such a system $\mathcal{B}$, which means it is of the form

$$\mathcal{B}^* \;:=\; \{N\setminus S \,:\, S\in\mathcal{B}\}\,.$$

We explain now that any min-balanced system $\mathcal{B}$ defines a unique standardized inequality (2) with $\alpha\in\mathbb{Z}^{\mathcal{P}(N)}$ such that $\mathcal{B}=\mathcal{B}_\alpha$.

3.1 How to assign an inequality to a min-balanced system

Given a min-balanced system $\mathcal{B}$, unique coefficients $\lambda_S>0$, $S\in\mathcal{B}$, exist with

$$\chi_M = \sum_{S\in\mathcal{B}} \lambda_S\cdot\chi_S \qquad \text{where } M=\bigcup\mathcal{B}.$$

In fact, one can even show that $\lambda_S\in\mathbb{Q}$. Indeed, one has $\chi_M=\sum_{S\in\mathcal{B}}\lambda_S\cdot\chi_S$ means that the coefficient vector $\lambda\in\mathbb{R}^{\mathcal{B}}$ is a solution of a matrix equality $\lambda\cdot C=\chi_M$ with a zero-one matrix $C\in\mathbb{R}^{\mathcal{B}\times N}$. Since a unique solution exists, a regular column $\mathcal{B}\times T$-submatrix of C, where $T\subseteq N$, $|T|=|\mathcal{B}|$, exists such that $\lambda\cdot C^{\mathcal{B}\times T}=\chi_{M\cap T}$. Since C has zero columns for $i\in N\setminus M$ one has $T\subseteq M$. Nevertheless, the inverse of this regular zero-one submatrix is a rational matrix, which implies that the components of λ are in $\mathbb{Q}$. Thus, a unique integer $k\geq 1$ exists such that $k\cdot\lambda_S\in\mathbb{Z}$, $S\in\mathcal{B}$, are relatively prime. One can put

$$\begin{aligned}
\alpha_{\mathcal{B}}(M) &:= k,\\
\alpha_{\mathcal{B}}(S) &:= -k\cdot\lambda_S \quad \text{for } S\in\mathcal{B},\\
\alpha_{\mathcal{B}}(\emptyset) &:= -\alpha_{\mathcal{B}}(M)-\sum_{S\in\mathcal{B}}\alpha_{\mathcal{B}}(S) = -k+k\cdot\sum_{S\in\mathcal{B}}\lambda_S,\\
\alpha_{\mathcal{B}}(R) &:= 0 \quad \text{for remaining } R\subseteq N.
\end{aligned}$$

It is shown in [7, § 3.1] that these coefficients define a standardized form of the inequality (2) and one has $\mathcal{B}_\alpha = \mathcal{B}$ with $\alpha = \alpha_{\mathcal{B}}$. This yields mutually inverse transformation between min-balanced systems and the coefficient vectors of ascribed inequalities.

4 Irreducible min-balanced systems

The next concept is related to the conjectures below.

Definition 4 We say that a min-balanced system $\mathcal{B} \subseteq \mathcal{P}(N)$ is *reducible* if there exists $X \subset M \equiv \bigcup \mathcal{B}$ and $Y \in \mathcal{B}_X := \{S \in \mathcal{B} : S \subset X\}$ such that

- χ_X is in the conic hull of $\{\chi_S : S \in \mathcal{B}_X\}$,
- χ_M is in the conic hull of $\{\chi_T : T \in \{X\} \cup \mathcal{B} \setminus \{Y\}\}$.

A min-balanced system $\mathcal{B} \subseteq \mathcal{P}(N)$ which is not reducible is called *irreducible*.
We say that a min-balanced system $\mathcal{B} \subseteq \mathcal{P}(N)$ is *weakly irreducible* if no set $X \subset \bigcup \mathcal{B}$ exists such that both $\mathcal{B}_X$ and $\{\bigcup \mathcal{B}_X\} \cup (\mathcal{B} \setminus \mathcal{B}_X)$ are min-balanced.

Note that, without loss of generality, one can only require $X = \bigcup \mathcal{B}_X \notin \mathcal{B}$ in the above definitions and the irreducibility implies the weak irreducibility; see [7, Observation 8]. The intended meaning of the irreducibility condition is that the inequality ascribed to $\mathcal{B}$ is not derivable from other inequalities for min-balanced systems $\mathcal{B}'$ where $\bigcup \mathcal{B}' \subseteq \bigcup \mathcal{B}$.

Here is an example of a reducible system.

Example Put $N = \{a, b, c, d\}$ and consider the set system $\mathcal{B} = \{\{a\}, \{b\}, \{c\}\}$, whose corresponding inequality is

$$m(abc) - m(a) - m(b) - m(c) + 2 \cdot m(\emptyset) \geq 0\,. \tag{5}$$

Take $X = \{a, b\}$ and observe that $\mathcal{B}_X = \{\{a\}, \{b\}\}$ is min-balanced; the same holds for $\mathcal{C} = \{X\} \cup (\mathcal{B} \setminus \mathcal{B}_X) = \{\{a, b\}, \{c\}\}$. Thus, $\mathcal{B}$ is not weakly irreducible, and therefore, not irreducible. The respective inequalities are

$$\begin{aligned} m(ab) - m(a) - m(b) + m(\emptyset) &\geq 0, \\ m(abc) - m(ab) - m(c) + m(\emptyset) &\geq 0, \end{aligned} \tag{6}$$

both facet-defining for $E(N)$. Clearly, (5) is the sum of the inequalities in (6).

An analogous procedure is possible for every reducible min-balanced system. The following result is shown in [7, Corollary 9].

Observation 5 Given a reducible min-balanced system $\mathcal{B}$, the corresponding inequality is a conic combination of inequalities which correspond to other min-balanced systems $\mathcal{B}'$ with $\bigcup \mathcal{B}' \subseteq \bigcup \mathcal{B}$.

In particular, the inequalities ascribed to reducible systems are never facet-defining for $T(N)$ or $E(N)$.

5 Conjectures

The first conjecture concerns the totally balanced cone.

Conjecture 1 *The facet-defining inequalities for $T(N)$ are just those ascribed to non-trivial irreducible min-balanced systems $\mathcal{B}$ on $M \subseteq N$, $|M| \geq 2$.*

Note that $T(N)$ is not closed under reflection; therefore, one cannot expect that a conjugate inequality to a facet-defining inequality is also facet-defining.

Conjecture 2 *The facet-defining inequalities for $E(N)$ are just those ascribed to non-trivial irreducible min-balanced systems $\mathcal{B}$ on $M \subset N$, $|M| \geq 2$, and the conjugate inequalities to these.*

Conjecture 2 is in line with the fact that $E(N)$ is closed under reflection. Note that the inequalities in Conjecture 2 imply inequalities ascribed to min-balanced systems on N. Moreover, if both conjectures are true, then one can derive easily that $m \in E(N)$ iff $m, m^* \in T(N)$. That would imply that a game m is exact iff both m and $\widetilde{m^*}$ are totally balanced.

6 Computations and examples

A former version of the conjectures was based on weak irreducibility concept. Therefore, in case $|N| \leq 8$, we have computed the permutational types of min-balanced and weakly irreducible min-balanced systems, respectively. We listed all min-balanced systems type representatives in a tree-like catalog with the following access keys (for example, take $\mathcal{B} = \{a, bd, cd, abc\}$):

- the number $|N|$ of players ($\mathcal{B}$: $|\{a, b, c, d\}| = 4$),
- the number of sets $|\mathcal{B}|$ in the system ($\mathcal{B}$: 4),
- ordered players' multiplicities $|\{B \in \mathcal{B} : i \in B\}|$, $i \in N$, ($\mathcal{B}$: $(2, 2, 2, 2)$),
- ordered cardinalities $|B|$, $B \in \mathcal{B}$, ($\mathcal{B}$: $(1, 2, 2, 3)$),
- ordered balancing coefficients λ_S, $S \in \mathcal{B}$, ($\mathcal{B}$: $(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$).

To recognize whether two given min-balanced systems are of the same type, we transformed the problem to the task of recognizing bipartite graph isomorphism. Specifically, players and sets are turned into graph nodes of two different parts. If a player is in a set, then an edge exists between the respective nodes. We have used BLISS algorithm [5], as implemented in *igraph* package [2] of R environment. To check whether a newly found min-balanced system is of a recorded permutational type (= already stored in the catalog), we searched through the leaves of the respective branch of the above tree only. Just one representative of each permutational class is stored in the catalog.

number of players	$n = 3$	$n = 4$	$n = 5$	$n = 6$	$n = 7$	$n = 8$
min-balanced types	3	9	40	428	15.309	1.597.581
weakly irreducible types	2	5	16	164	6.188	704.995

Table 1: Permutational types of non-trivial min-balanced systems on N.

To find whether the permutational type of min-balanced system is weakly irreducible, we applied Definition 4 directly. The above mentioned catalog of min-balanced systems was used to speed-up the computations.

The resulting numbers of permutation types for $3 \leq n = |N| \leq 8$ are shown in Table 1. Note that for $n \leq 4$ each weakly irreducible system is irreducible. In the case $|N| = 2$ only one non-trivial min-balanced system on N exists. It is irreducible and has the form $\mathcal{B} = \{a, b\}$ (2-partition).

In the case $n = |N| = 3$ five min-balanced systems exist which break into three permutational types; two of these types are irreducible:

- $\mathcal{B} = \{a, b, c\}$ represents a reducible type, 3-partition (1 system),
- $\mathcal{B} = \{a, bc\}$ represents an *irreducible type*, 2-partition (3 systems),
- $\mathcal{B} = \{ab, ac, bc\}$ represents an *irreducible type* (1 system).

In the case $n = |N| = 4$ one has 41 min-balanced systems which break into nine permutational types; five of these types are irreducible.

1. $\mathcal{B} = \{a, b, c, d\}$ represents a reducible type, 4-partition (1 system),
2. $\mathcal{B} = \{a, b, cd\}$ represents a reducible type, 3-partition (6 systems),
3. $\mathcal{B} = \{ab, cd\}$ represents an *irreducible type*, 2-partition (3 systems),
4. $\mathcal{B} = \{a, bcd\}$ represents an *irreducible type*, 2-partition (4 systems),
5. $\mathcal{B} = \{a, bc, bd, cd\}$ represents a reducible type (4 systems),
6. $\mathcal{B} = \{ab, acd, bcd\}$ represents an *irreducible type* (6 systems),
7. $\mathcal{B} = \{a, bd, cd, abc\}$ represents a reducible type (12 systems),
8. $\mathcal{B} = \{ab, ac, ad, bcd\}$ represents an *irreducible type* (4 systems),
9. $\mathcal{B} = \{abc, abd, acd, bcd\}$ represents an *irreducible type* (1 system).

Most of the above min-balanced systems are families of sets incomparable with respect to inclusion. Nonetheless, the min-balanced system in the 7-th item contains comparable sets a and abc.

Our computation also revealed irreducible min-balanced systems containing at least one pair of comparable sets: in case $N = \{a, b, c, d, e\}$ the system $\mathcal{B} = \{ab, acd, ace, bcde, abde\}$ is an irreducible min-balanced system.

7 Conclusions

Our future effort will be directed to the conjectures formulated in Section 5. The plan is to utilize the methods of polyhedral geometry either to confirm or to disprove them. The catalogues of irreducible min-balanced systems for $|N| = 6, 7, 8$ are highly useful in this context, because they determine the lists of inequalities conjectured to be facet-defining for $T(N)$ and $E(N)$.

In another direction of research related to the cones $\mathcal{S}(N)$ and $\mathcal{E}(N)$, we found simple linear criteria to recognize extreme supermodular/exact games and developed web platforms based on implementation of these criteria [14, 15].

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References

[1] O. Bondareva: Some applications of linear programming methods to the theory of cooperative games (in Russian). Problemy Kibernetiki 10 (1963) 119-139.

[2] G. Csardi, T. Nepusz: The igraph software package for complex network research, InterJournal Complex Systems 1695, 2006. Available through `http://igraph.org`.

[3] P. Csóka, P. J.-J. Herings, L. A. Kóczy: Balancedness condition for exact games. Mathematical Methods of Optimization Research 74 (2011) 44-52.

[4] M. Grabisch: Set Functions, Games and Capacities in Decision Making. Springer 2016.

[5] T. Junttila, P. Kaski: Engineering an efficient canonical labeling tool for large and sparse graphs. Proceedings of the 9-th on Algorithm Engineering and Experiments and the 4-th Workshop on Analytic Algorithms and Combinatorics, 2007, 135-149.

[6] J. Kuipers, D. Vermeulen, M. Voorneveld: A generalization of the Shapley-Ichiishi result. International Journal of Game Theory 39 (2010) 585-602.

[7] J. Kroupa, M. Studený: Towards facets of the exact cone. A manuscript in preparation (2018).

[8] E. Lohmann, P. Borm, P. J.-J. Herings: Minimal exact balancedness. Mathematical Social Sciences 64 (2012) 127-135.

[9] E. Miranda, I. Montes: Games solutions, probability transformations and the core. In JMLR Workshops and Conference Proceedings 62: ISIPTA 2017, 217-228.

[10] B. Peleg: An inductive method for constructing minimal balanced collections of finite sets. Naval Research Logistics Quaterly 12 (1965) 155-162.

[11] E. Quaeghebeur: Learning from samples using coherent lower probabilities. PhD thesis, Ghent University 2009. Available through `http://hdl.handle.net/1854/LU-495650`.

[12] L. S. Shapley: On balanced sets and cores. Naval Research Logistics Quarterly 14 (1967) 453-460.

[13] M. Studený: Probabilistic Conditional Independence Structures. Springer 2005.

[14] M. Studený, T. Kroupa: Core-based criterion for extreme supermodular functions. Discrete Applied Mathematics 20 (2016) 122-151.

[15] M. Studený, V. Kratochvíl: Linear core-based criterion for testing extreme exact games. In JMLR Workshops and Conference Proceedings 62: ISIPTA 2017, 313-324.