Decisions on generalized Anscombe-Aumann acts under possibly "unexpected" scenarios

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Abstract

We consider decisions on generalized Anscombe-Aumann acts, mapping states of the world to belief functions over a set of consequences. Preference relations on these acts are given by a decision maker under different scenarios (conditioning events). Then, we provide a system of axioms which are necessary and sufficient for the representability of these "conditional preferences" through a conditional functional $\mathbf{CEU}_{P,u}$, parametrized by a unique full conditional probability P on the algebra of events and a cardinal utility function u on consequences. The model is able to manage also "unexpected" (i.e., "null") conditioning events. We finally provide an elicitation procedure that reduces to a Quadratically Constrained Linear Problem (QCLP).

1 Introduction

In many decision problems under uncertainty in economics, we need to choose between uncertain consequences in a set X that are contingent on the states of the world in S. So, we distinguish between an "objective" uncertainty related to X (i.e., exogenously quantified and given to the decision maker, in the spirit of von Neumann-Mergenstern) and a "subjective" uncertainty related to S (i.e., encoded in the decision maker's preferences, in the spirit of Savage). This configures a two-stage process where first the state of the world is chosen by Nature, and then the consequence is chosen through "objective" uncertainty, in the spirit of [1].

Very often, due to partial knowledge, uncertainty cannot be encoded in a single probability measure, but we rather have a class of probability measures.

We refer to situations where ambiguity is related to the "objective" probabilistic assessment as that due to a partially known randomizing device (like an urn or a roulette wheel) that results in a class of probability measures whose lower envelope is a *belief function* [10, 27], like in the well-known Ellsberg's urn paradox [12]. Following [28], in these cases we will speak of "objective" ambiguity. Hence, the above objects of decisions can be modelled as generalized Anscombe-Aumann acts [1] mapping S to the set $\mathbf{B}(X)$ of belief functions over X, forming the set $\mathcal{F} = \mathbf{B}(X)^S$.

A crucial aspect of making decisions under uncertainty is the possibility of reasoning under hypotheses. Unexpected situations such as earthquakes, terror attacks or financial crises are normally identified with "null" events and are often ignored in decision problems. Nevertheless, "unexpected" scenarios can deeply impact on the analysis of a decision problem [19] and should not be discarded.

Here we consider a conditional decision model involving the above generalization of Anscombe-Aumann acts, assuming that the decision maker is able to provide a family of preference relations $\{\precsim_H\}_{H \in \wp(S)^0}$ on $\mathcal{F}$ indexed by the set $\wp(S)^0 = \wp(S) \setminus \{\emptyset\}$ of non-impossible events. Every preference relation $\precsim_H$ can be interpreted as comparing acts under the hypothesis H.

In the model we propose, "objective" ambiguity is expressed by referring to the class of belief functions over X (as in the models [4, 17]). On the other hand, "subjective" uncertainty is assumed to be probabilistic, so, we model it with a *full conditional probability* in the sense of [8, 11, 23], that allows for conditioning to "null" events, but possible.

Here, we search for a representation in terms of a conditional functional $\mathbf{CEU}_{P,u}$ parametrized by a full conditional probability $P(\cdot|\cdot)$ on $\wp(S) \times \wp(S)^0$ and a utility function $u : X \to \mathbb{R}$. The above conditional functional consists in a mixture with respect to a full conditional probability of Choquet expected utilities [4] contingent on the states of the world. In particular, due to the properties of the Choquet integral [25], every state-contingent Choquet expected utility is actually a lower expected utility with respect to the probabilities in $\mathbf{core}(f(s))$. The present model generalizes the conditional version of the Anscombe-Aumann model given in [21] by introducing "objective" ambiguity.

We provide a set of axioms for the family $\{\precsim_H\}_{H \in \wp(S)^0}$ that is proved to be necessary and sufficient for the existence of a unique full conditional probability $P(\cdot|\cdot)$ and a cardinal utility function u such that the corresponding $\mathbf{CEU}_{P,u}$ functional *represents* the preferences, i.e., for every $f, g \in \mathcal{F}$ and every $H \in \wp(S)^0$,

$$f \precsim_H g \iff \mathbf{CEU}_{P,u}(f|H) \leq \mathbf{CEU}_{P,u}(g|H).$$

It turns out that a rational agent in this model behaves as a $\mathbf{CEU}_{P,u}$ maximizer, so, as a maximizer of a conditional expected value of state-contingent lower expected utilities. Hence, the present model encodes a form of "objective" ambiguity aversion. The model can be easily extended in a way to cope with different attitudes towards "objective" ambiguity: this will be the subject of future research.

A similar decision setting, limited to the unconditional case, has been considered by [28], where the author takes acts mapping states of the world to non-empty compact convex polyhedral sets of probability measures over consequences. In the same paper the author considers a representation functional different from ours, but still relying on a mixture with respect to a "subjective" probability measure.

Important efforts have been addressed in the decision theory literature to model "subjective" ambiguity, that is to ambiguity in "subjective" uncertainty evaluations (see, e.g., the survey papers [13] and [15]). For instance, in the seminal papers [26] and [16], the classical Anscombe-Aumann setting is considered but there ambiguity is "subjective", since the mixture of state-contingent expected utilities is done through the Choquet integral with respect to a capacity over S in the first model, while a class of "subjective" probabilities is considered in the second model. Still working in the classical Anscombe-Aumann setting, we find the models [2, 3, 20]. Other lines of research take care of "subjective" ambiguity in a Savage's setting, through acts that map states of the world to non-empty sets of consequences [14, 22]. All the quoted decision models essentially focus on unconditional decisions.

The conditional functional $\mathbf{CEU}_{P,u}$ is completely specified once the full conditional probability $P(\cdot|\cdot)$ and the utility function u have been elicited by the decision maker. In general, an agent is only able to provide few comparisons for few conditioning events. In this case, the first issue is to check the consistency of the given comparisons with the model of reference. When consistency holds, it is easily seen that an elicitation procedure relying on a finite number of arbitrary comparisons cannot guarantee the uniqueness of P and the cardinality of u in general.

We provide an elicitation procedure that reduces to a Quadratically Constrained Linear Problem (QCLP). Unfortunately, the quadratic constraints in the problem are generally not positive definite, so, the problem is generally not convex: interior points algorithms are not suitable. The problem can be solved with a branch and bound algorithm coping with global optimization of non-linear problems, such as the Couenne optimizer [7].

2 Model description

Consider the following decision-theoretic setting:

- $X = \{x_1, \ldots, x_m\}$, a finite set of consequences;
- $\wp(X)^0 = \wp(X) \setminus \{\emptyset\}$, the set of multi-consequences, i.e., non-empty sets of consequences;
- $\mathbf{B}(X) = \{Bel : \wp(X) \to [0, 1]\}$, the set of all belief functions on $\wp(X)$;
- $S = \{s_1, \ldots, s_n\}$, a finite set of states of the world;
- $\wp(S)$, the set of events;

- $\wp(S)^0 = \wp(S) \setminus \{\emptyset\}$, the set of scenarios, i.e., non-impossible events;
- $\mathcal{F} = \mathbf{B}(X)^S = \{f : S \to \mathbf{B}(X)\}$, the set of all acts;
- $\{\precsim_H\}_{H \in \wp(S)^0}$, a family of preference relation on $\mathcal{F}$, indexed by the set of non-impossible events $H \in \wp(S)^0$.

For every $H \in \wp(S)^0$, we denote with $\prec_H$ and $\sim_H$ the asymmetric and symmetric parts of $\precsim_H$. Moreover, for every $f, g \in \mathcal{F}$, $f \precsim_H g$ means "f is not preferred to g under the hypothesis H", $f \prec_H g$ means "g is preferred to f under the hypothesis H", and $f \sim_H g$ means "f is indifferent to g under the hypothesis H".

Notice that the set $\mathbf{B}(X)$ contains the set

$$\mathbf{B}_0(X) = \{\delta_B \;:\; B \in \wp(X)^0\},$$

of *vacuous belief functions*, where δ_B is the belief function whose Möbius inversion is such that $m_{\delta_B}(B) = 1$ and 0 otherwise. Let us notice that $\mathbf{B}(X)$ is closed with respect to the *convex combination* operation defined, for every $Bel_1, Bel_2 \in \mathbf{B}(X)$ and every $\alpha \in [0,1]$, pointwise, for every $A \in \wp(X)$, as

$$(\alpha Bel_1 + (1-\alpha)Bel_2)(A) = \alpha Bel_1(A) + (1-\alpha)Bel_2(A),$$

and it holds

$$m_{\alpha Bel_1 + (1-\alpha)Bel_2} = \alpha m_{Bel_1} + (1-\alpha) m_{Bel_2}.$$

The set of acts $\mathcal{F}$ contains, in particular, the set of *constant acts* $\mathcal{F}_c$ whose elements are defined, for every $Bel \in \mathbf{B}(X)$, as

$$\overline{Bel}(s) = Bel, \;\forall s \in S.$$

The set $\mathcal{F}$ is closed with respect to the following operation of *convex combination*: for every $f, g \in \mathcal{F}$ and every $\alpha \in [0,1]$, $\alpha f + (1-\alpha)g$ is defined pointwise, for every $s \in S$, as

$$(\alpha f + (1-\alpha)g)(s) = \alpha f(s) + (1-\alpha)g(s).$$

For every $H \in \wp(S)^0$, the relation $\precsim_H$ determines a relation $\trianglelefteq_H$ on $\mathbf{B}(X)$ through constant acts defined, for every $Bel_1, Bel_2 \in \mathbf{B}(X)$, as

$$Bel_1 \trianglelefteq_H Bel_2 \Longleftrightarrow \overline{Bel_1} \precsim_H \overline{Bel_2}.$$

In turn, the relation $\trianglelefteq_H$ determines a relation $\leq^\bullet_H$ on $\wp(X)^0$ defined, for every $A, B \in \wp(X)^0$, as

$$A \leq^\bullet_H B \Longleftrightarrow \overline{\delta_A} \trianglelefteq_H \overline{\delta_B}.$$

Finally, the relation $\leq^\bullet_H$ induces a relation $\leq^*_H$ on X defined, for every $x, y \in X$, as

$$x \leq^*_H y \Longleftrightarrow \{x\} \leq^\bullet_H \{y\}.$$

Let $\leq^*$ be a weak order on X with asymmetric and symmetric parts $<^*$ and $=^*$, respectively, and assume $x_{\sigma(1)} \leq^* \ldots \leq^* x_{\sigma(m)}$, where σ is a permutation of $\{1, \ldots, m\}$. Then, denote $X^* = X_{/=^*} = \{[x_{i_1}], \ldots, [x_{i_t}]\}$ for which $<^*$ is a strict order, and we can assume $[x_{i_1}] <^* \cdots <^* [x_{i_t}]$. The $\leq^*$-*aggregated Möbius inversion* associated to $Bel \in \mathbf{B}(X)$ is the function $M_{Bel}^{\leq^*} : X^* \to [0,1]$ defined, for every $[x_{i_j}] \in X^*$, as

$$M_{Bel}^{\leq^*}([x_{i_j}]) = \sum_{x_i \in [x_{i_j}]} \sum_{x_i \in B \subseteq E_i^\sigma} m_{Bel}(B), \tag{1}$$

where $E_i^\sigma = \{x_{\sigma(i)}, \ldots, x_{\sigma(m)}\}$ for $i = 1, \ldots, m$. Note that $M_{Bel}^{\leq^*}([x_{i_j}]) \geq 0$ for every $[x_{i_j}] \in X^*$ and $\sum_{j=1}^t M_{Bel}^{\leq^*}([x_{i_j}]) = 1$, thus $M_{Bel}^{\leq^*}$ determines a probability distribution on X^*. It is easily seen that, if $u : X \to \mathbb{R}$ then defining $x \leq^* y$ if and only if $u(x) \leq u(y)$, for every $Bel \in \mathbf{B}(X)$, it holds

$$\oint u \mathrm{d}Bel = \sum_{[x_{i_j}] \in X^*} u(x_{i_j}) M_{Bel}^{\leq^*}([x_{i_j}]).$$

Let us stress that $M_{Bel}^{\leq^*}$ encodes a pessimistic aggregation of the uncertainty expressed by m_{Bel} [4]. Indeed, it holds

$$\sum_{[x_{i_j}] \in X^*} u(x_{i_j}) M_{Bel}^{\leq^*}([x_{i_j}]) = \sum_{B \in \wp(X)^0} \left(\min_{x \in B} u(x) \right) m_{Bel}(B).$$

We are searching for a representation of $\{\precsim_H\}_{H \in \wp(S)^0}$ in the form of a conditional mixture of Choquet integrals, i.e., for every $f \in \mathcal{F}$ and $H \in \wp(S)^0$,

$$\mathbf{CEU}_{P,u}(f|H) = \sum_{s \in S} P(\{s\}|H) \left(\oint u \mathrm{d}f(s) \right), \tag{2}$$

where $P(\cdot|\cdot)$ is a full conditional probability on $\wp(S) \times \wp(S)^0$ and $u : X \to \mathbb{R}$ is a cardinal utility function.

Consider the following axioms.

(AA1C) Weak order: $\forall H \in \wp(S)^0$, $\precsim_H$ is a weak order on $\mathcal{F}$;

(AA2C) Continuity: $\forall H \in \wp(S)^0$, $\forall f, g, h \in \mathcal{F}$, if $f \prec_H g \prec_H h$, $\exists \alpha, \beta \in (0,1)$ such that

$$\alpha f + (1-\alpha) h \prec_H g \prec_H \beta f + (1-\beta) h;$$

(AA3C) Independence: $\forall H \in \wp(S)^0$, $\forall f, g, h \in \mathcal{F}$ and $\forall \alpha \in (0,1)$

$$f \precsim_H g \Longleftrightarrow \alpha f + (1-\alpha) h \precsim_H \alpha g + (1-\alpha) h;$$

(AA4C) Monotonicity: $\forall H \in \wp(S)^0$, $\forall f, g \in \mathcal{F}$, if $f(s) \trianglelefteq_H g(s)$, $\forall s \in S$ then $f \precsim_H g$;

(AA5C) Non-triviality: $\forall H \in \wp(S)^0$, $\exists f, g \in \mathcal{F}$ such that $f \prec_H g$;

(AA6C) Relevance: $\forall H \in \wp(S)^0$, $\forall f, g \in \mathcal{F}$ with $f(s) = g(s)$, $\forall s \in H$ then $f \sim_H g$;

(AA7C) Uncertainty independence: $\forall f, g \in \mathcal{F}$ and $\forall H, K \in \wp(S)^0$, if $f \precsim_H g$, $f \precsim_K g$, and $H \cap K = \emptyset$ then $f \precsim_{H \cup K} g$;

(AA8C) State neutrality: $\forall s, t \in S$, if $f(s) = f(t)$, $g(s) = g(t)$, and $f \precsim_{\{s\}} g$ then $f \precsim_{\{t\}} g$.

(AA9C) Aggregate indifference: $\forall H \in \wp(S)^0$, $\forall A \in \wp(S)$ and $\forall f, g \in \mathcal{F}$ with $f(s) = g(s)$ $\forall s \in A$, if $M_{f(s)}^{\leq_H^*} = M_{g(s)}^{\leq_H^*}$ $\forall s \in A^c$ then $f \sim_H g$;

Axioms **(AA1C)**–**(AA5C)** are the usual Anscombe-Aumann axioms in the formulation of [26], stated for generalized Anscombe-Aumann acts and every preference relation in $\{\precsim_H\}_{H \in \wp(S)^0}$. Axioms **(AA6C)**–**(AA8C)** cope with conditioning. In particular, axiom **(AA6C)** expresses a focusing conditioning rule, i.e., it states that in conditioning to H, only the part of acts inside of H counts. Axiom **(AA7C)** copes with relating different conditioning events, while axiom **(AA8C)** encodes a form of consistency between different states. Finally, axiom **(AA9C)** is responsible for the $\mathbf{CEU}_{P,u}$ representation: it says that if two possibly distinct acts have the same $\leq_H^*$-aggregated Möbius inversion (i.e., the same pessimistic aggregation of "objective" uncertainty) then, they should be judged indifferent given H.

The following theorem, whose proof is omitted due to a lack of space, shows that axioms **(AA1C)**–**(AA9C)** are necessary and sufficient to get a $\mathbf{CEU}_{P,u}$ representation.

Theorem 1. *The following statements are equivalent:*

(i) the family of relations $\{\precsim_H\}_{H \in \wp(S)^0}$ *satisfies* **(AA1C)**–**(AA9C)***;*

(ii) there exist a full conditional probability $P : \wp(S) \times \wp(S)^0 \to [0,1]$ *and a non-constant utility function* $u : \wp(X)^0 \to \mathbb{R}$ *such that, for every* $f, g \in \mathcal{F}$ *and every* $H \in \wp(S)^0$*,* $f \precsim_H g \Longleftrightarrow \mathbf{CEU}_{P,u}(f|H) \leq \mathbf{CEU}_{P,u}(g|H)$*.*

Moreover, P *is unique and* u *is unique up to positive linear transformations.*

Let us stress that a $\mathbf{CEU}_{P,v}$ functional allows to take "null" (possible) conditioning events as hypotheses and, even more, it allows to order events in $\wp(S)^0$ according to their "unexpectation". For that, we define, for every $H, K \in \wp(S)^0$,

$$H \sqsubseteq K \Longleftrightarrow \mathbf{1}_\emptyset \prec_{H \cup K} \mathbf{1}_H,$$

with the meaning "H is no more unexpected than K", where the act $\mathbf{1}_E$, for $E \in \wp(S)$, is defined as in the proof of Theorem 1. The statement $H \sqsubseteq K$ expresses the uncertainty evaluation $P(H|H \cup K) > 0$, i.e., it considers the probability of the events H under the hypothesis that either H or K is true. In particular, $H \sqsubset K$ means $P(H|H \cup K) > 0$ and $P(K|H \cup K) = 0$, whereas $H =^{\square} K$ stands for $P(H|H \cup K) > 0$ and $P(K|H \cup K) > 0$. The relation $\sqsubseteq$ reveals to be a weak order on $\wp(S)^0$ and has been originally introduced by [9, 18, 24].

Every full conditional probability $P(\cdot|\cdot)$ on $\wp(S)$ is in bijection with a linearly ordered class of probability measure $\{P_0, \ldots, P_k\}$ on $\wp(S)$, said *complete agreeing class*, whose supports form a partition of S [5, 6].

Events with probability 0 essentially determine the structure of a full conditional probability $P(\cdot|\cdot)$ on $\wp(S)$ and actually the relation $\sqsubseteq$ is intimately related to $\{P_0, \ldots, P_k\}$.

Given $P(\cdot|\cdot)$, the corresponding complete agreeing class $\{P_0, \ldots, P_k\}$ representing it can be built through the events

$$H_0^\alpha = \{s \in H_0^{\alpha-1} \,:\, P(\{s\}|H_0^{\alpha-1}) = 0\} \quad \text{for } \alpha = 1, \ldots, k,$$

with $H_0^0 = S$, by setting $P_\alpha(\cdot) = P(\cdot|H_0^\alpha)$ with $H_0^\alpha \neq \emptyset$. On the other hand, given $\{P_0, \ldots, P_k\}$, for every $E|H \in \wp(S) \times \wp(S)^0$ there is a minimum index $\alpha_H \in \{0, \ldots, k\}$ such that $P_{\alpha_H}(H) > 0$ and it holds

$$P(E|H) = \frac{P_{\alpha_H}(E \cap H)}{P_{\alpha_H}(H)}.$$

The class of events $\{H_0^0, \ldots, H_0^k\}$ determines a decreasing class $\{\mathcal{I}_0, \ldots, \mathcal{I}_k\}$ of ideals of $\wp(S)$, singled out by the relation $\sqsubseteq$, defined as

$$\mathcal{I}_\alpha = \{A \in \wp(S)^0 \,:\, H_0^\alpha \sqsubseteq A\} \cup \{\emptyset\} = \{A \in \wp(S) \,:\, A \subseteq H_0^\alpha\}.$$

The class of events $\{H_0^0, \ldots, H_0^k\}$ also gives rise to a partition $\mathcal{E} = \{E_0, \ldots, E_k\}$ of S obtained by setting

$$E_\alpha = H_0^\alpha \setminus H_0^{\alpha-1} \quad \text{for } \alpha = 0, \ldots, k-1,$$

with $E_k = H_0^k$, where $E_\alpha = \mathbf{supp}(P_\alpha) = \{s \in S \,:\, P_\alpha(\{s\}) > 0\}$ in the complete agreeing class representing $P(\cdot|\cdot)$.

3 Model elicitation

The conditional functional $\mathbf{CEU}_{P,u}$ is completely specified once the full conditional probability $P(\cdot|\cdot)$ and the utility function u have been elicited by the decision maker (DM). In general, the DM is only able to provide few comparisons for few conditioning events. In this case, the first issue is to check the consistency of the given comparisons with the model of reference. When consistency holds, it is easily seen that an elicitation procedure relying on a finite number of arbitrary comparisons cannot guarantee the uniqueness of P and u in general.

Fixed X and S, we propose an elicitation procedure based on three different cognitive tasks.

We ask the DM to determine a subset $\mathcal{L} = \{H_1, \dots, H_N\} \subseteq \wp(S)$ that correspond to those events considered as "scenarios of interest" and then to order them according to their unexpectation, by providing a weak order $\sqsubseteq$ on $\mathcal{L}$.

We ask the DM to provide a weak order $\leq^*$ on X, i.e., on consequences obtained with certainty.

For every $H \in \mathcal{L}$, we ask the DM to provide a finite number of strict $\{f_l \prec_H g_l\}_{l \in L_H}$ and weak comparisons $\{f_w \precsim_H g_w\}_{w \in W_H}$, with $L_H \neq \emptyset$ while W_H is allowed to be empty. This assures non-triviality.

The issue is to find a complete agreeing class $\{P_0, \dots, P_k\}$ on $\wp(S)$ (and, so, a full conditional probability $P(\cdot|\cdot)$) *compatible* with the relation $\sqsubseteq$ on $\mathcal{L}$ (that is such that $H_i \sqsubseteq H_j \iff P(H_i|H_i \cup H_j) > 0$) and a utility function $u : X \to \mathbb{R}$ increasing with respect to $\leq^*$, such that the corresponding $\mathbf{CEU}_{P,u}$ conditional functional preserves all the strict and weak preference comparisons.

At this aim, let $\mathcal{L}_{/=_\Box} = \{[H_{i_1}], \dots, [H_{i_M}]\}$ and assume $[H_{i_1}] \sqsubset \dots \sqsubset [H_{i_M}]$. Now, define $B_0^{M+1} = \emptyset$ and for $\alpha = 0, \dots, M$, $B_0^\alpha = \bigcup_{\beta=\alpha}^M \bigcup_{H \in [H_{i_\beta}]} H$ and $E_0^\alpha = B_0^\alpha \setminus B_0^{\alpha+1}$.

Every linearly ordered class of probability measures $\{P_0^*, \dots, P_M^*\}$ on $\wp(S)$ where $\mathbf{supp}(P_\alpha^*) \subseteq E_0^\alpha$, for $\alpha = 0, \dots, M$, is said *minimal agreeing class* and determines a conditional probability $P^*(\cdot|\cdot)$ on $\wp(S) \times \mathbf{add}(\mathcal{L})$, where $\mathbf{add}(\mathcal{L})$ is the set of events obtained closing $\mathcal{L}$ with respect to unions. The conditional probability $P^*(\cdot|\cdot)$ can be further extended (generally not in a unique way) to a full conditional probability $P(\cdot|\cdot)$ on $\wp(S)$ *compatible* with $\sqsubseteq$ on $\mathcal{L}$. One of the possible extensions is determined by the *complete agreeing class* $\{P_0^*, \dots, P_M^*, P_{M+1}^*\}$ where P_{M+1}^* is an arbitrary probability measure on $\wp(S)$ such that $\mathbf{supp}(P_{M+1}^*) = S \setminus \bigcup_{\alpha=0}^M \mathbf{supp}(P_\alpha^*)$. The adjunct of P_{M+1}^* is necessary only if $S \setminus \bigcup_{\alpha=0}^M \mathbf{supp}(P_\alpha^*) \neq \emptyset$.

With such an input, the elicitation procedure consists in solving the following optimization problem with unknowns the minimal agreeing class $\{P_0^*, \dots, P_M^*\}$, the utility function u and the dummy variable δ:

$$\text{maximize } \delta \text{ subject to:}$$

$$\begin{cases}
\displaystyle\sum_{s \in E_0^\alpha} P_\alpha^*(\{s\}) \left(\sum_{[x_{i_j}] \in X^*} u(x_{i_j}) \left(M^{\leq^*}_{f_l(s)}([x_{i_j}]) - M^{\leq^*}_{g_l(s)}([x_{i_j}]) \right) \right) + \delta \leq 0, \\
\displaystyle\sum_{s \in E_0^\alpha} P_\alpha^*(\{s\}) \left(\sum_{[x_{i_j}] \in X^*} u(x_{i_j}) \left(M^{\leq^*}_{f_w(s)}([x_{i_j}]) - M^{\leq^*}_{g_w(s)}([x_{i_j}]) \right) \right) \leq 0, \\
\displaystyle\sum_{s \in E_0^\alpha} P_\alpha^*(\{s\}) = 1, \\
P_\alpha^*(\{s\}) \geq 0, \ \forall s \in E_0^\alpha, \\
u(x_{i_1}) = 0, \ u(x_{i_t}) = 1, \ u(x_{i_j}) - u(x_{i_{j+1}}) + \delta \leq 0, \ \text{for } j = 1, \dots, t-1, \\
-1 \leq \delta \leq 1,
\end{cases}$$

for $\alpha = 1, \ldots, M$ and all $H \in [H_{i_\alpha}]$, for all $l \in L_H$, for all $w \in W_H$. The above optimization problem is a Quadratically Constrained Linear Problem (QCLP) that is a particular case of a Quadratically Constrained Quadratic Problem (QCQP). Unfortunately, the quadratic constraints in the problem are generally not positive definite, so, the problem is generally not convex: interior points algorithms are not suitable. The problem can be solved with a branch and bound algorithm coping with global optimization of non-linear problems, such as the COUENNE optimizer [7].

Solving the above optimization problem allows to check both the consistency of the given preference statements and, if consistency holds, to find a full conditional probability $P(\cdot|\cdot)$ and a utility function u determining the conditional functional $\mathbf{CEU}_{P,u}$. Indeed, the preference statements are consistent with the model if and only if $\delta > 0$ and in this case the solution of the system determines $P(\cdot|\cdot)$ and u, up to the possible arbitrary choice of the probability measure P^*_{M+1}.

4 A paradigmatic example

Take the set of states of the world $S = \{s_1, s_2, s_3, s_4\}$ spanned by events

- K = "North Korea and USA enter into war next year";
- G = "Italian GDP increases next year";

with $K = \{s_1, s_2\}$ and $G = \{s_1, s_3\}$.

Consider three unitary financial instruments that can result in a loss of €50, in a null gain or in a gain of €100, implying $X = \{-50, 0, 100\}$. From statistics of previous years we only have partial information on the performances of each instrument, that are listed below:

Instrument 1: It is only known that it guarantees a gain of €100 in 30% of cases;

Instrument 2: It is only known that it results in a loss of €50 in 20% of cases;

Instrument 3: No information is available.

Hence, instrument i determines a class of probability measures $\mathbf{P}^i$ on $\wp(X)$ whose lower envelope is easily shown to be a belief function $Bel_i = \min \mathbf{P}^i$, with:

$$\begin{aligned}
\mathbf{P}^1 &= \{P : \wp(X) \to [0,1] | P \text{ is a probability measure}, \gamma \in [0, 0.7], \\
&\qquad P(\{-50\}) = \gamma, P(\{0\}) = 0.7 - \gamma, P(\{100\}) = 0.3\}, \\
\mathbf{P}^2 &= \{P : \wp(X) \to [0,1] | P \text{ is a probability measure}, \gamma \in [0, 0.8] \\
&\qquad P(\{-50\}) = 0.2, P(\{0\}) = \gamma, P(\{100\}) = 0.8 - \gamma\}, \\
\mathbf{P}^3 &= \{P : \wp(X) \to [0,1] | P \text{ is a probability measure}\}.
\end{aligned}$$

Consider the following investment strategies in which the adopted financial instrument is contingent on the state of the world:

	s_1	s_2	s_3	s_4
f	Bel_3	Bel_1	Bel_1	Bel_2
g	Bel_3	Bel_3	Bel_2	Bel_3

The question is: How should a DM decide between f and g conditionally to events K and K^c?

Suppose that our DM is not able to express directly his preference between f and g, conditionally to K and K^c. Nevertheless, our DM is a profit maximizer and believes that a war between North Korea and USA next year is unexpected, while it is more likely a decrease of Italian GDP next year.

The fact that event K is unexpected, i.e., it is judged as "null" by our DM, does not rule out its possible realization. In particular, if event K were true then our DM believes that it would be more likely an increase of Italian GDP, due to a profit of Italian weapons factories.

Hence, our DM is able to provide the following information: $\mathcal{L} = \{K, K^c\}$ with $K^c \sqsubset K$; $-50 <^* 0 <^* 100$; the worst and best multi-consequences $\underline{A} = \{-50\}$ and $\overline{A} = \{100\}$. For every $E \in \wp(S)$, define the act

$$\mathbf{1}_E(s) = \begin{cases} \delta_{\overline{A}}, & \text{if } s \in E, \\ \delta_{\underline{A}}, & \text{if } s \notin E. \end{cases}$$

In turn, the beliefs of our DM can be translated as follows:

$$\mathbf{1}_{\{s_3\}} \prec_{K^c} \mathbf{1}_{\{s_4\}} \quad \text{and} \quad \mathbf{1}_{\{s_2\}} \prec_K \mathbf{1}_{\{s_1\}}.$$

In this case we have that $E_0^0 = K^c$ and $E_0^1 = K$. To avoid cumbersome notation, denote $p_i^\alpha = P_\alpha^*(\{s_i\})$ and $u_1 = u(-50), u_2 = u(0), u_3 = u(100)$. We need to solve the following optimization problem

$$\text{maximize } \delta \text{ subject to:}$$
$$\begin{cases} -p_3^0 u_1 + p_3^0 u_3 + p_4^0 u_1 - p_4^0 u_3 + \delta \leq 0, \\ p_1^1 u_1 - p_1^1 u_3 - p_2^1 u_1 + p_2^1 u_3 + \delta \leq 0, \\ p_3^0 + p_4^0 = 1, \\ p_3^0, p_4^0 \geq 0, \\ p_1^1 + p_2^1 = 1, \\ p_1^1, p_2^1 \geq 0, \\ u_1 = 0,\ u_2 = 1,\ u_1 - u_2 + \delta \leq 0,\ u_2 - u_3 + \delta \leq 0, \\ -1 \leq \delta \leq 1, \end{cases}$$

for which the COUENNE optimizer finds the solution $p_3^0 = 0.18358$, $p_4^0 = 0.81642$, $p_1^1 = 0.81642$, $p_2^1 = 0.18358$, $u_1 = 0$, $u_2 = 0.5$, $u_3 = 1$, and $\delta = 0.5$. Since $\delta > 0$ the preference statements are consistent with the model and a full conditional probability $P(\cdot|\cdot)$ on $\wp(S)$ is that represented by the complete agreeing class $\{P_0^*, P_1^*\}$ whose distributions are

	$\{s_1\}$	$\{s_2\}$	$\{s_3\}$	$\{s_4\}$
P_0^*	0	0	0.18358	0.81642
P_1^*	0.81642	0.18358	0	0

With such $P(\cdot|\cdot)$ and u we have

$$\begin{aligned}\mathbf{CEU}_{P,u}(g|K) = 0 \quad &< \quad 0.055074 = \mathbf{CEU}_{P,u}(f|K),\\ \mathbf{CEU}_{P,u}(g|K^c) = 0.073432 \quad &< \quad 0.381642 = \mathbf{CEU}_{P,u}(f|K^c),\end{aligned}$$

so, $g \prec_K f$ and $g \prec_{K^c} f$, i.e., under both hypothesis the DM should choose f.

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