About Two Consonant Conflicts of Belief Functions

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Abstract

General belief functions usually bear some internal conflict which comes mainly from disjoint focal elements. Analogously, there is often some conflict between two (or more) belief functions. After the recent observation of hidden conflicts (seminar CJS'17 [17]), appearing at belief functions with disjoint focal elements, importance of interest in conflict of belief functions has increased.

This theoretical contribution introduces a new approach to conflicts (of belief functions). Conflicts are considered independently of any combination rule and of any distance measure.

Consonant conflicts are based on consonant approximations of belief functions in general; two special cases of the consonant approach based on consonant inverse pignistic and consonant inverse plausibility transforms are discussed.

Basic properties of the newly defined conflicts are presented, analyzed and briefly compared with our original approaches to conflict (combinational conflict, plausibility conflict and comparative conflict), with the recent conflict based on non-conflicting parts, as well as with W. Liu's degree of conflict.

1 Introduction

Belief functions (BFs; introduced in [25]) are one of the widely used formalisms for uncertainty representation and processing - that enables representation of incomplete and uncertain knowledge, belief updating, and combination of evidence.

Complications with highly conflicting belief functions combination, see e.g., [9, 28], have motivated a theoretical investigation of conflicts between belief functions [2, 4, 11, 18, 21, 22, 23, 24]. The problematic issue of an essence of conflict between belief functions - originally defined by the non-normalised version of Dempster's rule $\textcircled{\cap}$ (i.e., by its value for the empty set: $m_{\textcircled{\cap}}(\emptyset)$) - was first mentioned by Almond [1], and discussed further by W. Liu [22]. Almond's counter-example has been overcome by W. Liu's progressive approach. Unfortunately, the substance of the issue has not been solved there as positive conflict still may be detected for non-conflicting BFs.

Further steps ahead were presented in our previous study [11] where new ideas concerning interpretation, definition, and measurement of conflicts of BFs were introduced. Three new approaches to interpretation and computation of conflicts were suggested: combinational conflict, plausibility conflict (see also [13, 14]), and comparative conflict; pignistic conflict analogous to plausibility one was defined later in [14]. Unfortunately, none of those captures the nature of conflict sufficiently enough ands these approaches need further elaboration. Nevertheless, the very important distinction between conflict of two BFs and the internal conflict of an individual BF was pointed out in [11] - altogether with the necessity to distinguish between a conflict and a difference/distance of two BFs; this was also pointed out in [3].

Probabilistic approximations of belief functions were used in several previous approaches, e.g. pignistic transform in W. Liu's two-dimensional degree of conflict [22] and in pignistic conflict [14], normalized plausibility of singletons in plausibility conflict [11, 13, 14], etc.

Unfortunately, application of a probability approximation adds a new additional information, which increases internal conflict of inputs and also resulting in a global conflict. The new reverse approach suggested in this paper adds no new information but removes an information creating the internal conflicts, as inverse probabilistic transformations are used to make consonant approximations. This is an analogy to belief discounting, but without necessity of any parameter due to its specific context. Thus BFs without internal conflicts are used for a computation of a conflict; it is a generalization of the approach from [15] in fact.

2 Preliminaries

2.1 General Primer on Belief Functions

We assume classic definitions of basic notions from theory of *belief functions* [25] on a finite frame of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. A *basic belief assignment (bba)* is a mapping $m : \mathcal{P}(\Omega) \longrightarrow [0,1]$ such that $\sum_{A\subseteq\Omega} m(A) = 1$; the values of the bba are called *basic belief masses (bbm)*. $m(\emptyset) = 0$ is usually assumed - then we speak about *normalized bba*. A *belief function (BF)* is a mapping $Bel : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Bel(A) = \sum_{\emptyset\neq X\subseteq A} m(X)$. A *plausibility function* $Pl : \mathcal{P}(\Omega) \longrightarrow [0,1]$, $Pl(A) = \sum_{\emptyset\neq A\cap X} m(X)$. There is a unique correspondence among m and corresponding Bel and Pl thus we often speak about m as about belief function.

A *focal element* is a subset X of the frame of discernment, such that $m(X) > 0$. Let $\mathcal{F} = \{X \mid m(X) > 0\}$ be the set of all focal elements; and *core* be its union $\mathcal{C} = \bigcup_{X\in\mathcal{F}} X$. If all the focal elements are *singletons* (i.e. one-element subsets of Ω), then we speak about a *Bayesian belief function* (BBF). If all the focal elements are either singletons or whole Ω (i.e. $|X| = 1$ or $|X| = |\Omega|$), then we speak about a *quasi-Bayesian belief function* (qBBF). If all focal elements have non-empty intersections, we call this a *consistent belief function.* And if all focal elements are nested, we call this a *consonant belief function.* *Vacuous BF* (VBF) has the only focal element Ω: $m_{vac}(\Omega) = 1$. A *symmetric BF* is a BF, which has the same bbms for focal elements with the same cardinality, i.e., $m(X) = m(Y)$ for $|X| = |Y|$.

Let us recall *normalized plausibility of singletons*[1] of Bel: the BBF (probability distribution) $Pl\_P(Bel)$ such, that $(Pl\_P(Bel))(\omega_i) = \frac{Pl(\{\omega_i\})}{\sum_{\omega \in \Omega} Pl(\{\omega\})}$ [5, 10]; and alternative Smets' *pignistic probability*[2] $BetP(\omega_i) = \sum_{\omega_i \in X} \frac{m(X)}{|X|}$ [27].

2.2 A Graphical Presentation of Sets of Belief Functions

We can represent any BF on an n-element frame of discernment Ω_n by an enumeration of its m values (bbms), i.e., by a (2^n-2)-tuple $(x_1, x_2, ..., x_{2^n-2})$ as $m(\emptyset) = 0$ and $m(\Omega) = x_{2^n-1} = 1 - \sum_{i=1}^{2^n-2} x_i$. Thus we can present set of all BFs on Ω_n by a (2^n-2)-dimensional simplex in general. Specially we have 2D triangle and 6D simplex for Ω_2 and Ω_3, see Figure 1 [19, 20] and Figure 2 [12].

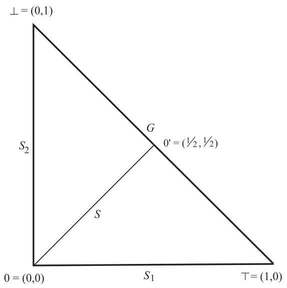

Figure 1: Belief functions on 2-element frame Ω_2; G: Bayesian BFs, S: symmetric BFs, S_1, S_2: simple support BFs ($\sim$ consonant BFs on Ω_2).

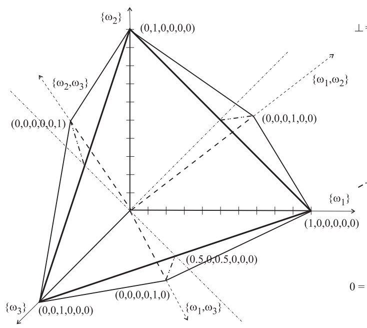

Figure 2: Simplex of Belief functions on 3-element frame Ω_3. 6 dimensions corresponds to 6 possible focal elements.

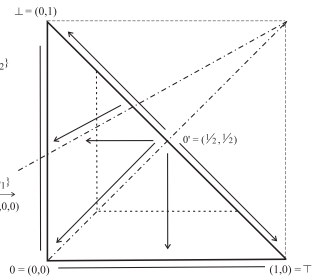

Figure 3: Internal conflict *Pl-IntC* on Ω_2. It has max value $\frac{1}{2}$ for $0'$, decreases along arrows, constant along lines without arrows; zero at S_i's.

3 Conflicts of Belief Functions

Conflicts of belief functions are caused mainly by disjoint focal elements either within individual BFs or in different BFs. *Internal conflicts* $IntC(m_i)$ of individual BFs are distinguished from *conflict between BFs* $Conf(m_1, m_2)$ [11]; the entire sum of multiples of mutually conflicting masses is called *total conflict* $TotC(m_1, m_2)$[3].

[1] $Pl\_P(Bel)$ is a normalization of contour function (of plausibility of singletons [25]) in fact.

[2] We have to note an analogy between pignistic probability and Shapley value [26].

[3] Some authors (see e.g. [18]) use 'total conflict' for maximal possible conflict, which arises when all focal elements of one BF are disjoint with focal element of another BFs. Our total

3.1 Internal Conflict of Belief Functions

Internal conflict of a BF is caused either by its disjoint focal elements (if there are any), or if $Pl(\{\omega\}) < 1$ for every $\omega \in \Omega$ (i.e. $\forall\omega \in \Omega$ exists focal element X_ω such that $omega \notin X_\omega$). Let us accept the following simple definition of internal conflict[4]: *Internal conflict of BF Bel* is defined by formula $IntC(Bel) = 1 - max_{\omega\in\Omega}Pl(\{\omega\})$, where Pl is the plausibility corresponding to Bel. A BF Bel is *(internally) non-conflicting* when it has zero internal conflict $IntC(Bel) = 0$; it is *(internally) conflicting* otherwise. This definition corresponds to internal plausibility conflict $Pl\text{-}IntC(Bel)$ from [11], see Section 4.1.

Thus a BF is non-conflicting if and only if there is an $\omega \in \Omega$ such that $Pl(\{\omega\}) = 1$ (or in other words if BF is consistent).

3.2 Conflicts between Belief Functions

There are several different assumptions about conflicts between belief functions in our previous approaches [11, 13, 14, 15]. Some of them are mutually conflicting as coming from various alternative approaches, thus we suppose only those common:

A1. Non-negativity and boundary conditions: $0 \leq Conf(Bel_1, Bel_2) \leq 1$.
A2. Symmetry: $Conf(Bel_1, Bel_2) = Conf(Bel_2, Bel_1)$.
A3. $Conf(Bel, Bel) = 0$. A BF is not conflicting with itself.
A4. $Conf(Bel, Bel_{vac}) = 0$. Vacuous BF is non-conflicting with any other BF.

The other assumptions in our previous approaches are stronger and they distinguish among various approaches. Thus we do not consider them among our general assumptions here. We may compare our assumptions with Martin's axioms MA1 – MA5 [23] and Destercke & Burger properties P3 – P6 [18]:

(MA1) : $Conf(Bel', Bel'') \geq 0$,
(MA2) : $Conf(Bel, Bel) = 0$,
(MA3) : $Conf(Bel', Bel'') = Conf(Bel'', Bel')$,
(MA4) : $Conf(Bel', Bel'') \leq 1$,
(MA5) : $Conf(Bel', Bel'') = 0$ iff $m' \subseteq m$" or $m'' \subseteq m'$ [5].

(P3) : Extreme values: $Conf(Bel_1, Bel_2) = 0$ iff $\bigcap_{X\in\mathcal{F}_1\cup\mathcal{F}_2} \neq \emptyset$ iff
iff $\sum_{X\cap Y=\emptyset} m_1(X)m_2(Y) = 0)$ iff $\mathcal{P}_{m_1} \cap \mathcal{P}_{m_2} \neq \emptyset$,
where $\mathcal{P}_m = \{Prob \mid Bel(X) \leq Prob(A),\ \forall X \subseteq \Omega\}$;
$Conf(Bel_1, Bel_2) = 1$ iff $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.
(P4) : Symmetry.
(P5) : Imprecision monotonicity.
(P6) : 'Ignorance is Bliss' $\sim Conf(Bel, Bel_{vac}) = 0$.

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conflict correspond to their global conflict.

[4] Let us note, that there are different approaches how to define internal conflict of BFs, e.g., using minnimal entropy functional, see [2], or using author's conflicting parts of BFs [12].

[5] A very special case of belief specialization: $m' \subseteq m$" implies that m' is a specialization of m'', but reverse implication does not hold true.

(P7) : Insensitivity to refinement. (a conflict should not be changed if frame of discernment is refined)

A1 corresponds to axioms MA1 and MA4 and it is consistent with properties P3, P5, P6. A2 corresponds to MA3 and to property P4. A3 corresponds to axiom MA2, this is not assumed by D & B, on the other hand it inconsistent with strong property P3. A4 follows rather strong MA5; A4 corresponds to P6.

4 Former Approaches to Conflict between BFs

To compare the new consonant approach to conflict, let us briefly introduce the former approaches.

4.1 Three Approaches from IPMU 2010

Unfortunately, there are not yet any precise formulas[6], but only bounding inequalities for *combinational conflicts*: $\frac{1}{2}TotC(m,m)) \leq IntC(m) \leq TotC(m,m)$, $TotC(m_1,m_2)-(IntC(m_1)+IntC(m_2)) \leq C(m_1,m_2) \leq TotC(m_1,m_2)$.

Internal plausibility conflict of BF Bel is defined as $Pl\text{-}IntC(Bel) = 1 - max_{\omega\in\Omega} Pl(\{\omega\})$, where Pl is the plausibility equivalent to Bel.

Plausibility conflict between BFs Bel_1, Bel_2 is defined by the formula $Pl\text{-}C(Bel_1, Bel_2) = min(\sum_{\omega\in\Omega_{PlC}(Bel_1,Bel_2)} \frac{1}{2}|Pl\_P(Bel_1) - Pl\_P(Bel_2)(\omega)|, (m_1 \circledcirc m_2)(\emptyset))$,where $\Omega_{PlC}(Bel_1, Bel_2)$ is the set of elements $\omega \in \Omega$ with conflicting Pl_P masses [11, 14].

The idea of comparative conflictness/non-conflictness is a specification of bbms to smaller focal elements such that fit to focal elements of the other BF as much as possible. *The comparative conflict between BFs* Bel_1 *and* Bel_2 is defined as the least difference of such more specified bbms derived from the input m_1 and m_2.

4.2 Liu's Degree of Conflict and Pignistic Conflict

The above 3 approaches were compared with Liu's *degree of conflict* cf in [11]; cf is defined as $cf(m_i, m_j) = (m_\oplus(\emptyset),\ difBetP_{m_i}^{m_j})$ in [22], where $m_\oplus(\emptyset)$ should be rather $m_{\circledcirc}(\emptyset)$ (more precisely $(m_i \circledcirc m_j)(\emptyset)$) in fact, $difBetP_{m_i}^{m_j}$ is defined as $difBetP_{m_i}^{m_j} = max_{A\subseteq\Omega}(|BetP_{m_i}(A) - BetP_{m_j}(A)|)$. It holds: $difBetP_{m_i}^{m_j} = Diff(BetP_{m_i}, BetP_{m_j}) = \frac{1}{2}\sum_{\omega\in\Omega} |BetP_{m_i}(\{\omega\}) - BetP_{m_j}(\{\omega\})|$ [13].

Pignistic conflict is an alternative of the plausibility conflict [14], where pignistic probability $BetP$ is used instead of normalised plausibility of singletons.

4.3 Conflict Based on Non-Conflicting Parts

For the recent measure of conflict *ncp-Conf* [15] is based on Daniel's ideas from [11] and namely from [12]. When analysing properties of approaches from [11] using

[6] Let us recall that notion 'total conflict' $TotC$ is used for global conjunctive conflict $GlcC$ [11].

Hájek-Valdés algebraic approach [19, 20], hypothesis[7] of decomposition of a BF into its conflicting and non-conflicting parts was formulated in [12]; and existence of unique non-conflicting part Bel_0 of any BF Bel was proven there:

Theorem 1. *Let $h(Bel) = Bel \oplus U_n$, where U_n is the uniform distribution on sigletons, i.e., $U_n(\{\omega_i\}) = \frac{1}{n}$. For any BF Bel defined on Ω_n there exists unique consonant BF Bel_0 such that, $h(Bel_0 \oplus Bel_S) = h(Bel)$ for any BF Bel_S such that $Bel_S \oplus U_n = U_n$.*

Definition 1. *Let Bel', Bel'' be two belief functions on n-element frame of discernment $\Omega_n = \{\omega_1, \omega_2, ..., \omega_n\}$. Let Bel'_0 and Bel''_0 be their non-conflicting parts and m'_0, m''_0 the related bbas. We define* conflict between BFs Bel' and Bel'' *as* $ncp\text{-}Conf(Bel', Bel'') = m_{Bel'_0 \circledcirc Bel''_0}(\emptyset) = (m'_0 \circledcirc m''_0)(\emptyset)$. *Where $\circledcirc$ is non-normalised Dempster's (conjunctive) rule of BFs combination.*

For algorithm of computation of $ncp\text{-}Conf(Bel', Bel'')$ see [15].

5 Consonant Conflicts between BFs

Probabilistic approximations of belief functions were used in several previous approaches, e.g. pignistic probability in Liu's degree of conflict cf and in pignistic conflict $BetP\text{-}C$, and normalized plausibility of singletons in plausibility conflict $Pl\_P\text{-}C$.

Making a probabilistic approximation has two disadvantages in general: the approximation adds some additional information and internal conflict of a BF is increased. As we do not know how internal conflicts of individual BFs participates in global conflict of these BFs, a probabilistic approximation brings also an unspecified contribution to the conflict "between" which is defined using the transformation.

Our present idea is to use consonant approximations $cAppox(Bel)$ instead of the probabilistic ones. Theoretically, we can use any consonant approximation such that the original BF is its specialization. We use more strict condition: inverse probabilistic transformations, i.e., such that $Transf(Bel) = Transf(cAppox(Bel))$, specially for pignistic transformation $BetT(Bel) = BetP$ and plausibility (i.e. contour) transform $Pl\_T(Bel) = Pl\_P$. Thus for $BetT(Bel) = BetP$ we will use *consonant pignistic inverse* $iBetT(Bel) =_{iBet} Bel = iBet$, given by $_{iBet}m$, i.e., consonant inverse of $BetP$: $BetT(iBetT(Bel)) = BetT(Bel) = BetP$ and *consonant inverse contour* iC, i.e., consonant inverse of Pl_P: $Pl\_T(iCT(Bel)) = Pl\_T(iPl\_T(Bel)) = Pl\_T(Bel) = Pl\_P$, see Figure 4.

These approximations have several advantages: they have no internal conflict - the entire conflict is the conflict "in between". No additional information nor internal conflict is added; internally conflicting information is removed. Analogously

[7]This hypothesis have been proven only on BFs on Ω_2 in [12]. In general case, the conflicting part seems to be in a close relationship to internal conflict of the BF.

to the original probabilistic approximations these are also uniquely defined and probabilistic approximation is preserved.

Having consonant approximations of two belief functions, we define a conflict between them. As the internal conflict of consonant approximations is zero, the entire conflict of these approximations is the conflict "in between". Let us adopt the simplest sum of multiples of disjoint focal elements:

Definition 2. *Let Bel_1, Bel_2 be two belief functions on Ω, ${}_{iC}Bel_i = iCT(Bel_i)$ and ${}_{iBet}Bel_i = iBetT(Bel_i)$ be their consonant inverse contour and consonant inverse pignistic approximations given by consonant bbas ${}_{iC}m_i$, ${}_{iBet}m_i$.*
Inverse contour conflict *is defined by formula*

$$iC\text{-}Conf(Bel_1, Bel_2) = \sum_{X \cap Y = \emptyset} {}_{iC}m_1(X)_{iC}m_2(Y),$$

where $X, Y \subseteq \Omega$ (i.e., where $X \in \mathcal{F}_{{}_{iC}m_1}$, $Y \in \mathcal{F}_{{}_{iC}m_2}$).
Inverse pignistic conflict *is analogously defined by*

$$iBet\text{-}Conf(Bel_1, Bel_2) = \sum_{X \cap Y = \emptyset} {}_{iBet}m_1(X)_{iBet}m_2(Y),$$

where $X, Y \subseteq \Omega$ (i.e., where $X \in \mathcal{F}_{{}_{iBet}m_1}$, $Y \in \mathcal{F}_{{}_{iBet}m_2}$).

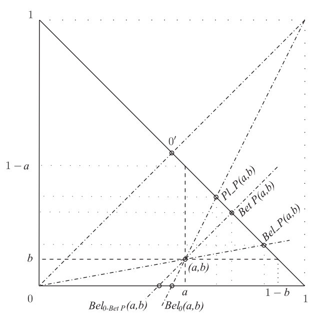

Figure 4: Consonant Approximations on Ω_2

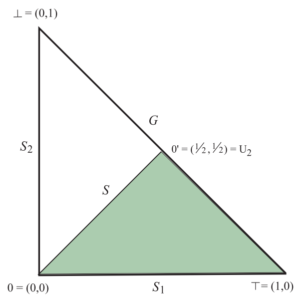

Figure 5: Simplices of mutually non-conflicting Belief functions on Ω_2.

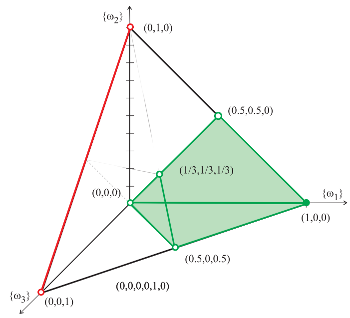

Figure 6: Simplex of quasi Bayesian BFs non-conflicting with $(1, 0, 0)$.

5.1 Basic Properties of iC-$Conf$ and $iBet$-$Conf$

We may easily verify that Definition 2 satisfy assumptions A1 – A4.

Lemma 1. *The following is equivalent: (i) $iC\text{-}Conf(Bel_1, Bel_2) = 0$*
(ii) $\bigcap_{X \in \mathcal{F}_{{}_{iC}m_i}} X \neq \emptyset$
(iii) $X_0 \cap Y_0 \neq \emptyset$ *where* $X_0 \in \mathcal{F}_{{}_{iC}m_1}, Y_0 \in \mathcal{F}_{{}_{iC}m_2}$ *and*
$X_0 \subset X$, $Y_0 \subset Y$ *for any* $X \in \mathcal{F}_{{}_{iC}m_1}, Y \in \mathcal{F}_{{}_{iC}m_2}$
(iv) $\{\omega_M | PlP_1(\omega_M) \geq PlP_1(\omega), \omega \in \Omega\} \cap \{\omega_M | PlP_1(\omega_M) \geq PlP_2(\omega), \omega \in \Omega\} \neq \emptyset$
(v) $\{\omega_M | Pl_1(\omega_M) \geq Pl_1(\omega), \omega \in \Omega\} \cap \{\omega_M | Pl_1(\omega_M) \geq Pl_2(\omega), \omega \in \Omega\} \neq \emptyset.$

Lemma 2. *The following is equivalent: (i) iBet-Conf*$(Bel_1, Bel_2) = 0$
(ii) $\bigcap_{X\in\mathcal{F}_{iBet}m_i} X \neq \emptyset$
(iii) $X_0 \cap Y_0 \neq \emptyset$ *where* $X_0 \in \mathcal{F}_{iBet}m_1$, $Y_0 \in \mathcal{F}_{iBet}m_2$ *and*
$X_0 \subset X$, $Y_0 \subset Y$ *for any* $X \in \mathcal{F}_{iBet}m_1, Y \in \mathcal{F}_{iBet}m_2$
(iv) $\{\omega_M | BetP_1(\omega_M) \geq BetP_1(\omega), \omega\in\Omega\} \cap \{\omega_M | BetP_1(\omega_M) \geq BetP_2(\omega), \omega\in\Omega\} \neq \emptyset$.

Lemma 3. *For any pair of BFs on* Ω_2 *and, generally, for any pair of qBBFs* Bel_1 Bel_2 *on* Ω_n *it holds that*
(i) iC-Conf$(Bel_1, Bel_2) = 0$ *iff iBet-Conf*$(Bel_1, Bel_2) = 0$ *iff*
$\{\omega \mid m_1(\{\omega\}) = max_i\{m_1(\{\omega_i\})\}\} \cap \{\omega \mid m_2(\{\omega\}) = max_i\{m_2(\{\omega_i\})\}\} \neq \emptyset$,
i.e., if $(a_1 - b_1)(a_2 - b_2) \geq 0$ *in the case of* $Bel_i = (a_i, b_i)$ *on* Ω_2*;*
(ii) iC-Conf$(Bel_1, Bel_2) \geq$ *iBet-Conf*(Bel_1, Bel_2).

From the last condition of Lemma 3 (i) it follows that the following holds in our graphical presentation: any two BFs on Ω_2 from right hand half of the triangle are mutually non-conflicting (there is no conflict between them, see the green part of the triangle on Figure 5; analogously for any BFs from left hand white part.

Analogously, it holds for qBBFs on Ω_n: any two qBBFs from an n-dimensional subsimplex (1/n of the entire simplex of qBBFs, which is defined by $(0, 0, 0, ..., 0)$, and corresponding segment of BBFs where $m(\{\omega^*\}) \geq m(\{\omega\})$ for a given $\omega^* \in \Omega_n$ i.e., $1/n$ of $(n-1)$-dimensional subsimplex of BBFs including $m_{\{\omega^*\}} : m_{\{\omega^*\}}(\{\omega^*\}) = 1$) are mutually non-conflicting. E.g. on Ω_3 and $m_{\{\omega_1\}}(\{\omega_1\}) = 1$ and segment of BBFs given by $(1, 0, 0), (\frac{1}{2}, \frac{1}{2}, 0), (\frac{1}{2}, 0, \frac{1}{2})$, and $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, see green subsimplex on Figure 6. Any BF from green subsimplex is conflicting with any BFs from the white part (the rest) of the simplex, i.e. there is some positive conflict between them. For categorical qBBF $(1, 0, 0)$, the maximal value 1 of conflict appears with any BF from red line between $(0, 0, 1)$ and $(0, 1, 0)$. The green subsimplex of mutually non-conflicting BFs is one of maximal consistent simplices discussed in [6].

Corollary 1. *Any symmetric qBBF* Bel_S *on* Ω_n *is non-conflicting with any other qBBF Bel, i.e., iC-Conf*$(Bel_S, Bel) = 0 =$ *iBet-Conf*(Bel_S, Bel).

The situation is significantly more complicated on a $2^{(n-2)}$-dimensional simplex of general BFs on Ω_n. There is multidimensional structure of BFs instead of 1-dimensional *h-line* (a straight line[8] connecting a BF and related Pl_P); i.e., multidimensional structure of BFs with the same Pl_P. Analogously there is a multidimensional structure of BFs with the same $BetP$ instead of simple 1-dimensional perpendicular in the case of qBBFs. A simplex of BFs non-conflicting with $(1, 0, 0, 0, 0, 0)$ analogous to that of Figure 6 is 6-dimensional for general BFs on Ω_3. Thus we have no simple generalization of Lemma 3 to general BFs. On the other hand we can generalize its Corollary to Lemma 4.

Example 1. Let $m_1 = (1, 0, 0, 0, 0, 0)$; for $m_2 = (\frac{1}{3}, 0, 0, 0, 0, \frac{2}{3})$ we obtain $BetP_2 = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}, 0, 0, 0) = U_3$, ${}_{iBet}m_2 = (0, 0, 0, 0, 0, 0) = 0$ thus it is *iBet*-non-conflicting

[8] Its name comes from homomorhism h of algebraic structure of BFs, which is defined by $h(Bel) = Bel \oplus U_n$, see [8, 20].

with m_1; whereas $PlP_2 = (\frac{1}{5}, \frac{2}{5}, \frac{2}{5}, 0, 0, 0)$, ${}_{iC}m_2 = (0, 0, 0, 0, 0, \frac{2}{3})$, there is $iC\text{-}Conf$ $(m_1,m_2) = \frac{2}{3} > 0 = iBet\text{-}Conf(m_1,m_2)$. On the other hand, for $m_3 = (0,0,\frac{1}{2},\frac{1}{2},0,0)$ we obtain $PlP_3 = U_3$ thus ${}_{iC}m_2 = 0$, whereas $BetP_3 = (\frac{1}{4}, \frac{1}{4}, \frac{1}{2}, 0, 0, 0)$, and ${}_{iBet}m_3 = (0, 0, \frac{1}{4}, 0, 0, 0)$, hence $iC\text{-}Conf(m_1, m_3) = 0 < \frac{1}{4} = iBet\text{-}Conf(m_1, m_3)$.

Lemma 4. *(i) Any general symmetric BF Bel_S on Ω_n is non-conflicting with any other BF Bel, i.e., $iC\text{-}Conf(Bel_S, Bel) = 0 = iBet\text{-}Conf(Bel_S, Bel)$.*
(ii) Any BF Bel_{UPl} such that its $PlP_{UPl} = U_n$ is iC-non-conflicting with any BF Bel on Ω_n, i.e., for any Bel and any Bel_{UPl} with uniform plausibility it holds that $iC\text{-}Conf(Bel_{UPl}, Bel) = 0$.
(iii) Any BF Bel_{UBet} such that its $BetP_{UBet} = U_n$ is iBet-non-conflicting with any BF Bel on Ω_n, i.e., for any Bel and any Bel_{UBet} with uniform BetP holds $iBet\text{-}Conf(Bel_{UBet}, Bel) = 0$.

Lemma 5. *For any BF Bel on Ω_n, its core $\mathcal{C}_m$, cores $\mathcal{C}_{PlP}$, $\mathcal{C}_{BetP}$ of related probabilities and cores $\mathcal{C}_{{}_{iC}m}$, $\mathcal{C}_{{}_{iBetP}m}$ of their consonant approximations hold that $\mathcal{C}_m = \mathcal{C}_{PlP} = \mathcal{C}_{{}_{iC}m} = \mathcal{C}_{BetP} = \mathcal{C}_{{}_{iBet}m}$.*

Lemma 6. *For any pair of BFs Bel_1, Bel_2 on Ω_n and their cores $\mathcal{C}_{m_1}$, $\mathcal{C}_{m_2}$ it holds that (i) $iC\text{-}Conf(Bel_1, Bel_2) = 1$ iff $\mathcal{C}_{m_1} \cap \mathcal{C}_{m_2} = \emptyset$,*
(ii) $iBet\text{-}Conf(Bel_1, Bel_2) = 1$ iff $\mathcal{C}_{m_1} \cap \mathcal{C}_{m_2} = \emptyset$.

5.2 An Equivalence of Consonant *iC-Conf* to Conflict between BFs Based on their Non-Conflicting Parts *ncp-Conf*

Lemma 7. *Consonant inverse contour conflict $iC\text{-}Conf$ is equivalent to conflict between belief functions based on their non-conflicting parts $Conf$, i.e., for any pair of BFs Bel', Bel'' on Ω_n it holds that $iC\text{-}Conf(Bel', Bel'') = ncp\text{-}Conf(Bel', Bel'')$.*

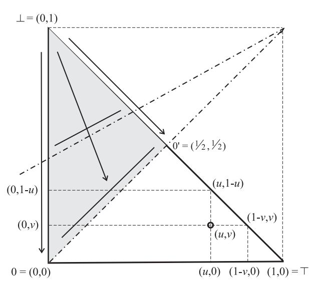

Figure 7: Inverse contour conflict between fixed BF (u, v) and general BF (a, b) on Ω_2; $iC\text{-}Conf((u, v), (a, b))$ decreases in direction of arrows and it is constant along lines without arrows.

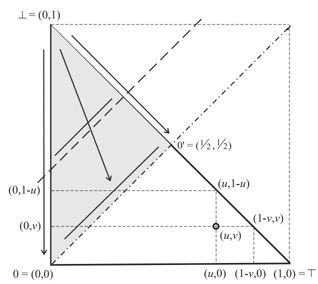

Figure 8: Inverse pignistic conflict between fixed BF (u, v) and general BF (a, b) on Ω_2; $iBet\text{-}Conf((u, v), (a, b))$ decreases in direction of arrows and it is constant along lines without arrows.

Unfortunately, we have found general counterexamples against Theorem 2 from [15], thus the following holds only for qBBFs, it does not hold in general.

Theorem 2. *(i) Let Bel_1 and Bel_2 be arbitrary quasi Bayesian BFs on general finite frame of discernment Ω_n given by bbas m' and m''. For both conflicts iC-Conf and iBet-Conf between Bel_1 and Bel_2 it holds that*

$$Conf(Bel_1, Bel_2) \leq \sum_{X \cap Y = \emptyset} m_1(X) m_2(Y).$$

(ii) Equality $Conf(Bel_1,Bel_2) = \sum_{X \cap Y = \emptyset} m_1(X)m_2(Y)$ holds iff both BFs Bel_1 and Bel_2 are consonant.

This statement does not hold for general BFs. see Example 2 [17].

Example 2. (Counter-example against Theorem 2 from Belief'14 [15] on Ω_3) Let us suppose Ω_3, $m_1(\{\omega_1, \omega_2\}) = 0.7$, $m_1(\{\omega_1, \omega_3\}) = 0.3$, and $m_2(\{\omega_2, \omega_3\}) = 1.0$. There is $Pl_1 = (1.0, 0.7, 0.3, ...)$, $iC_1 = (0.3, 0, 0, 0.4, 0, 0)$, $Pl_2 = (0, 1.0, 1.0, ...)$, $iC_2 = (0, 0, 0, 0, 0, 1.0)$, thus $iC\text{-}Conf(m_1, m_2) = 0.3 \cdot 1.0 = 0.3$; analogously $BetP_1 = (0.5, 0.35, 0.15)$, $iBet_1 = (0.15, 0, 0, 0.4, 0, 0)$, $BetP_2 = (0, 0.5, 0.5)$, $iBet_2 = (0, 0, 0, 0, 0, 1.0)$, thus $iBet\text{-}Conf(m_1, m_2) = 0.15$.
Nevertheless $\sum_{X \cap Y = \emptyset} m_1(X)m_2(Y) = 0 < 0.15 = iBet\text{-}Conf(m_1, m_2) < 0.30 = iC\text{-}Conf(m_1, m_2)$.

5.3 Relationships to Axiomatic Approaches

We have already seen that Martins axioms MA1 – MA4 are satisfied, due to their correspondence with our satisfied assumptions A1 – A3. MA5 is not satisfied as it is too strong due to Martin's strong definition of bba inclusion, nevertheless our assumption A4 is a consequence of MA5. Martin explicitly does not assume triangle inequality $Conf(Bel',Bel''') \leq Conf(Bel',Bel'') + Conf(Bel'',Bel''')$. Both consonant conflicts are the cases, where triangle inequality does not hold true, see Ex. 3.

Example 3. Let $Bel' = (0.4,0.1,0.1,0.2,0,0.1;0.1)$, $Pl' = (\frac{7}{15}, \frac{5}{15}, \frac{3}{15})$, $Bel'' = (0.3,0.2,0.1,0.1,0,0.1;0.2)$, $Pl'' = (\frac{6}{16}, \frac{6}{16}, \frac{4}{16})$, $Bel''' = (0.1,0.2,0.3,0.1,0,0.2;0.1)$, $Pl''' = (\frac{3}{15}, \frac{6}{15}, \frac{6}{15})$, $Bel'_0 = (\frac{2}{7}, 0, 0, \frac{2}{7}, 0, 0; \frac{3}{7})$, $Bel''_0 = (0, 0, 0, \frac{2}{6}, 0, 0; \frac{4}{6})$, $Bel'''_0 = (0, 0, 0, 0, 0, \frac{3}{6}, 0, 0; \frac{3}{6})$, $iC\text{-}Conf(Bel', Bel''') = \frac{1}{7} \not\leq 0 + 0 = iC\text{-}Conf(Bel', Bel'') + iC\text{-}Conf(Bel'', Bel''')$.

P3: We have equivalence only for maximal value; the strongest minimal value condition (i) implies (consonant) non-conflictness in general, medium condition (ii) does it only for quasi Bayesian BFs, and the weakest condition (iii) does not imply non-conflictness at all; either one of reverse implications does not hold true (either for qBBFs). We have validity of P4 and and P6, see our assumptions A2 and A4 above. P5 does not hold either for one of consonant conflicts as specialization of bbms can change order of plausibility and $BetP$ values, thus also focal elements of consonant approximations. P7 is most interesting of the properties, it distinguishes consonant conflicts: it holds for $iC\text{-}Conf$ whereas does not hold for $iBet\text{-}Conf$, due to that plausibility and iC approximations are consistent with refinement of the frame of discernment, but $BetP$ and $iBet$ do not.

Theorem 3. *Let Bel_1, Bel_2 be any BFs given by m_1, m_2 on general Ω_n. For both consonant conflicts iC-Conf and iBet-Conf between Bel_1 and Bel_2 it holds that (i) if $\bigcap_{X\in\mathcal{F}_1\cup\mathcal{F}_2} = \emptyset$ then $Conf(Bel_1, Bel_2) = 0$, (ii) if both Bel_1 and Bel_2 are quasi Bayesian and $\sum_{X\cap Y=\emptyset} m_1(X)m_2(Y)=0$ then $Conf(Bel_1, Bel_2)=0$, (iii) $Conf(Bel_1, Bel_2) = 1$ iff $\mathcal{C}_1 \cap \mathcal{C}_2 = \emptyset$.*

5.4 A Comparison with Previous Approaches to Conflict

5.4.1 Combinational Conflict

We suppose $TotC(m_1, m_2) - (IntC(m_1) + IntC(m_2)) \leq C(m_1, m_2) \leq TotC(m_1, m_2)$ for combinational conflict. On the other hand, $0 \leq Conf(Bel_1, Bel_2) \not\leq \sum_{X\cap Y=\emptyset} m_1(X)m_2(Y) = TotC(Bel_1, Bel_2)$ for both *iC-Conf* and *iBet-Conf*. Thus both the consonant conflicts are not compatible with the formulation of combinational conflict [11].

5.4.2 Plausibility Conflict *Pl-C*

More interesting is a comparison with the most elaborated and precisely defined plausibility conflict. We can observe that: (*Conf* stands for *iC-Conf* or *iBet-Conf*)

Lemma 8. *For any couple of belief functions (a, b), (c, d) on 2-element frame of discernment it holds that: (i) $Conf((a, b), (c, d)) = 0$ iff $Pl\text{-}C((a, b), (c, d)) = 0$, (ii) $Conf((a, b), (c, d)) \leq Pl\text{-}C((a, b), (c, d))$.*
For any couple of belief functions Bel', Bel'' on general finite frame of discernment Ω_n it holds that: (iii) $iC\text{-}Conf(Bel', Bel'')=0$ iff $Pl\text{-}C_{sm}(Bel', Bel'')=0$.
(iv) If $Pl\text{-}C_0(Bel', Bel'')=0$ then also $iC\text{-}Conf(Bel', Bel'') = 0$ (in general; but not for final $Pl\text{-}C(Bel', Bel'')$).
(v), (vi) For qBBFs (iii) + (iv) hold also for iBet-Conf.

Thus, the nature of $Conf((a, b), (c, d))$ is very close to that of $Pl\text{-}C((a, b), (c, d))$. $Conf((a, b), (c, d))$ is simpler as its conflictness/non-conflictness simply comes from $\sum_{X\cap Y=\emptyset} {}_{iC}m_1(X)_{iC}m_2(Y) = \frac{|a-b|}{1-min(a,b)} \cdot \frac{|d-c|}{1-min(c,d)}$, $\sum_{X\cap Y=\emptyset} {}_{iBet}m_1(X)_{iBet}m_2(Y) = |a-b| \cdot |d-c|$. Hence there is no necessity to check conflictness of all focal elements. For the same nature see also Figures 7 and 8 which fit also to *Pl-C* and *Bet-C*, respectively.

5.4.3 Comparative Conflict *cp-C*

Comparative conflict has a completely different nature. There are mutually comparatively conflicting couples of BFs with same *max Pl* or *max BetP* elements of Ω_n. Thus they are non-conflicting according to *Conf*, (e.g., $Bel' = (0.5, 0.3, 0, 0, 0, 0.1)$, $Bel'' = (0.7, 0, 0, 0, 0.1, 0.1)$.) On the other hand, there are comparatively non-conflicting BFs, which prefer different ω_i's - they are conflicting according to *Conf* (e.g. $Bel' = (0.4, 0.2, 0.1, 0.1, 0, 0)$, $Bel'' = (0.3, 0.4, 0.1, 0.1, 0, 0)$). *cp-C* has some relationship to property P5, which should be investigated in future.

5.4.4 Liu's Measure of Conflict cf

Any couple of BFs that is mutually non-conflicting according to cf (Section 4.2) is (under some conditions) also mutually non-conflicting according to consonant conflicts (the reverse does not hold true); and $Conf$ is less or equal to cf. For behaviour of values of both the components of $cf = (m_{\circledcap}(\emptyset), difBetP)$ of a fixed (u,v) with a general (a,b) on Ω_2 see Figures 9, 10; values of both the components decrease in direction of arrows, they are constant along lines without arrow.

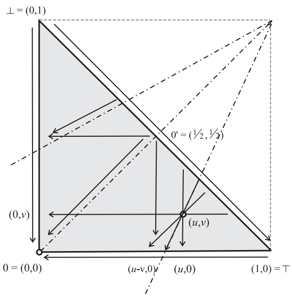

Figure 9: $m_{\circledcap}(\emptyset)$ component of cf between fixed (u,v) and general (a,b).

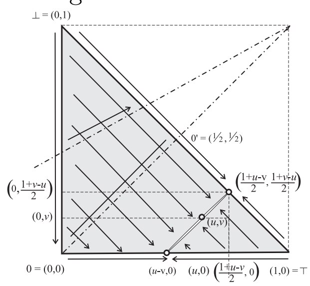

Figure 10: $difBetP$ between fixed BF (u,v) and general BF (a,b) on Ω_2.

Lemma 9. *(i) For any couple of belief functions Bel', Bel'' on n-element frame of discernment it holds that, if* $cf(Bel', Bel'') = ((m' \circledcap m'')(\emptyset), difBet^{Bel''}_{Bel'}) = (0,0)$ *then also* $iBet\text{-}Conf(Bel', Bel'') = 0$*,*
(ii) The above holds also for any pair of qBBFs Bel', Bel'' and $iC\text{-}Conf(Bel', Bel'')$
(iii) For $Bel' = (a,b)$, $Bel'' = (c,d)$ on Ω_2 holds further: $Conf((a,b),(c,d)) \leq cf((a,b),(c,d))$*; or precisely* $Conf \leq ((a,b) \circledcap (c,d))(\emptyset)$ & $Conf \leq difBet^{(c,d)}_{(a,b)}$.

6 Conclusion

In this study, we introduced a new approach of conflict between belief functions on general finite frame of discernment. Properties of its instances *iC-Conf* and *iBet-Conf* were analyzed and compared with former approaches. Conflict based on non-conflicting parts of BFs [15] was observed to be equivalent to consonant conflict *iC-Conf*. Further, satisfaction of Martin's [23] and Destercke-Burger's [18] axioms was studied.

A common elaboration of the theoretic principles of the presented results with those from [18] and [23] is a challenge for a future research. It should include an analysis of positive conflict in situations such that $\sum_{X \cap Y = \emptyset} m_1(X) m_2(X) = 0$.

The presented theoretical results improve general understanding of conflict between belief functions and entire nature of belief functions. Correct understanding of conflicts may, consequently, improve a combination of conflicting belief functions.

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